

# Faster Small-Constant-Periodic Merging Networks

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## Abstract

We consider the problem of merging two sorted sequences on a comparator network that is used repeatedly, that is, if the output is not sorted, the network is applied again using the output as input. The challenging task is to construct such networks of small depth (called a period in this context). In our previous paper *Faster 3-Periodic Merging Network* we reduced twice the time of merging on 3-periodic networks, i.e. from  $12\log N$  to  $6\log N$ , compared to the first construction given by Kutylowski, Loryś and Oesterdikhoff. Note that merging on 2-periodic networks require linear time. In this paper we extend our construction, which is based on Canfield and Williamson  $(\log N)$ -periodic sorter, and the analysis from that paper to any period  $p \geq 4$ . For  $p \geq 4$  our  $p$ -periodic network merges two sorted sequences of length  $N/2$  in at most  $\frac{2p}{p-2}\log N + p\frac{p-8}{p-2}$  rounds. The previous bound given by Kutylowski et al. was  $\frac{2.25p}{p-2.42}\log N$ . That means, for example, that our 4-periodic merging networks work in time upper-bounded by  $4\log N$  and our 6-periodic ones in time upper-bounded by  $3\log N$  compared to the corresponding  $5.67\log N$  and  $3.8\log N$  previous bounds. Our construction is regular and follows the same periodification schema, whereas some additional techniques were used previously to tune the construction for  $p \geq 4$ . Moreover, our networks are also periodic sorters and tests on random permutations show that average sorting time is closed to  $\log^2 N$ .

**Keywords:** parallel merging, oblivious merging, comparison networks, merging networks, periodic networks, comparators

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## 1 Introduction

Comparator networks are probably the simplest, comparison-based parallel model that is used to solve such tasks as sorting, merging or selecting [1]. Each network represents a data-oblivious algorithm, which can be easily implemented in other parallel models and hardware. Moreover, sorting networks can be applied in secure, multi-party computation (SMC) protocols. They are also used to encode cardinality constraints to propositional formulas [2] and are strongly connected with switching networks [3]. The most famous constructions of sorting networks are Odd-Even and Bitonic networks of depth  $\frac{1}{2}\log^2 N$  due to Batcher [4] and AKS networks of depth  $O(\log N)$  due to Ajtai, Komlos and Szemerédi [5]. The long-standing disability to decrease a large constant hidden behind the asymptotically optimal complexity of AKS networks to a practical value has resulted in studying easier, sorting-related problems, whose optimal networks have small constants. For a review on merging networks and sorting network see, for example, Knuth [1].

A comparator network consists of a set of  $N$  registers, each of which can store an item from a totally ordered set, and a sequence of comparator stages. Each stage is a set of comparators that connect disjoint pairs of registers and, therefore, can work in parallel (a comparator is a simple device that takes a contents of two registers and performs a compare-exchange operation on them: the minimum is put into the first register and the maximum into the second one). Stages are run one after another in synchronous manner, hence we can consider the number of stages as the running time. The size of a network is defined to be the total number of comparators in all its stages.

A network  $A$  consisting of stages  $S_1, S_2, \dots, S_d$  is called  $p$ -periodic if  $p < d$  and for each  $i$ ,  $1 \leq i \leq d - p$ , stages  $S_i$  and  $S_{i+p}$  are identical. A periodic network can be easier to implement, because one can use the

first  $p$  stages in a cycle: if the output of  $p$ -th stage is not correct (sorted, for example), the sequence of  $p$  stages is run again. In pure oblivious context, such computations are stopped after a predefined number of passes. We can also define a  $p$ -periodic network just by giving the total number of stages and a description of its first  $p$  stages. A challenging task is to construct a family of small-periodic networks for sorting-related problems with the running time equal to, or not much greater than that of non-periodic networks.

Dowd et al. [6] gave the construction of  $\log N$ -periodic sorting networks of  $N$  registers with running time of  $\log^2 N$ . Bender and Williamson introduced a large class of such networks [7]. Kutyłowski et al. [8] introduced a general method to convert a non-periodic sorting network into a 5-periodic one, but the running time increases by a factor of  $O(\log N)$  during the conversion. For simpler problems such as merging or correction there are constant-periodic networks that solve the corresponding problem in asymptotically optimal logarithmic time [9, 10, 11]. In particular, Kutyłowski, Loryś and Oesterdikhoff [9] have given a description of 3-periodic network that merges two sorted sequences of  $N$  numbers in time  $12\log N$  and a similar network of period 4 that works in  $5.67\log N$ . They sketched also a construction of merging networks with periods larger than 4 and running time decreasing asymptotically to  $2.25\log N$ . Note that 2-periodic merging networks require linear time.

In this paper we extend our construction from [12] of a new family of 3-periodic merging networks, which is based on Canfield and Williamson  $(\log N)$ -periodic sorter [13], and the underlying analysis to any period  $p \geq 4$ . For  $p \geq 4$  our  $p$ -periodic network merges two sorted sequences of length  $N/2$  in at most  $\frac{2p}{p-2}\log N + p\frac{p-8}{p-2}$  rounds. The previous bound given by Kutyłowski et al. [9] was  $\frac{2.25p}{p-2.42}\log N$ . That means, for example, that our 4-periodic merging networks work in time upper-bounded by  $4\log N$  and our 6-periodic ones in time upper-bounded by  $3\log N$ , compared to the corresponding  $5.67\log N$  and  $3.8\log N$  previous bounds. Our construction is regular and follows the same periodification schema as we used for 3-periodic merging networks, whereas some additional techniques were used previously to tune the construction for  $p \geq 4$ . Increasing  $p$  further, the multiplicative constant decreases approaching 2. The construction is pretty simple, but its analysis is quite complicated.

The advantage of constant-periodic networks is that they have pretty simple patterns of communication links, that is, each node (register) of such a network can be connected only to a constant number of other nodes. Such patterns are easier to implement, for example, in hardware. Moreover, a node uses these links in a simple periodic manner and this can save control logic and simplify timing considerations. We can also easily implement an early stopping property with  $p$ -periodic networks: if none of the comparators exchanged values in the last  $p$  stages, we could stop the computation. Since our networks are also periodic sorters, we have used this property to measure sorting times on random permutations and the results are quite surprising: the average sorting time of  $N$  items is closed to  $\log^2 N$ . Results are presented in Section 4.

The paper is organized as follows. In Section 2 we introduce a new periodification scheme, define our new family of  $p$ -periodic merging networks and give the main theorem. Section 3 is devoted to its proof, where we order the set of registers into a matrix and analyse the behaviour of our network by tracing the numbers of ones in its columns.

## 2 Periodic merging networks

Our merging networks are based on the Canfield and Williamson [13]  $(\log N)$ -periodic sorters. In the following proposition we recall the definition of the networks and their merging/sorting properties (see also Fig. 1). Recall that  $[i : j]$  denotes a comparator connecting registers  $i$  and  $j$ .

**Proposition 1.** (see [13]) *For  $k \geq 1$  let  $S_1 = \{[2i : 2i+1] : i = 0, 1, \dots, 2^{k-1} - 1\}$  and for  $j = 1, \dots, k-1$  let  $S_{j+1} = \{[2i+1 : 2i+2^{k-j}] : i = 0, 1, \dots, 2^{k-1} - 2^{k-j-1} - 1\}$ . Let  $CW_k = S_1, \dots, S_k$  be a network of  $N_k = 2^k$  registers numbered  $0, \dots, N_k - 1$ . Then (i) if two sorted sequences of length  $2^{k-1}$  are given in registers with odd and even indices, respectively, then  $CW_k$  is a merging network; (ii)  $CW_k$  is a  $k$ -pass periodic sorting network.*

We would like to implement a version of this network as a  $p$ -periodic comparator network. We begin with the definition of an intermediate construction  $P_k^p$  which structure is similar to the structure of  $CW_k$ . Then we transform it to  $p$ -periodic network  $M_k^p$ . Observe that in any  $N$ -register merging network we must have all *short* comparators  $[i : i+1]$ ,  $0 \leq i < N-1$ , and consecutive short comparators  $[i-1 : i]$  and  $[i : i+1]$

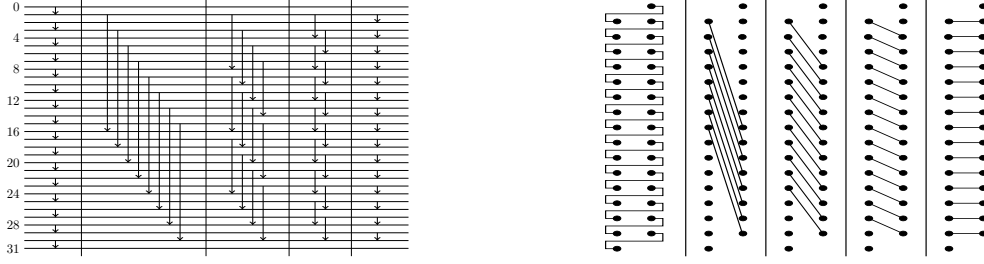


Figure 1: The Canfield and Williamson  $(\log N)$ -periodic sorter  $CW_5$ , where  $N = 32$ . On the left side, registers and comparators are represented by horizontal lines and arrows, respectively. On the right side, registers and comparators are represented by dots and edges, respectively. Stages are separated by vertical lines.

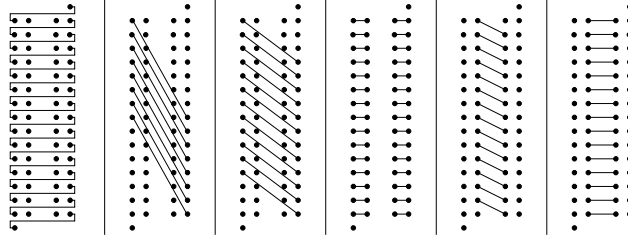


Figure 2:  $P_5^4$  as an implementation of  $CW_5$ . Registers and comparators are represented by dots and edges, respectively. Stages are separated by vertical lines. Stages with short horizontal comparators are inserted between stages with long comparators.

must be in different stages. The idea is to replace each register  $i$  in  $CW_k$  (except the first and the last ones) with a sequence of  $\lceil \frac{k-2}{p-2} \rceil$  consecutive registers, move the endpoints of  $i$ -th group of  $p-2$  long comparators one register further or closer depending on the parity of  $i$  and insert between each group of stages containing long comparators a stage with short comparators joining the endpoints of those long ones. The result is depicted in Fig. 2. In this way, we obtain a network in which each register is used in at most  $p$  consecutive stages. Therefore the network  $P_k^p$  can be packed into the first  $p$  stages and used periodically to get the desired  $p$ -periodic merging network.

A comparator  $[i : j]$  is *standard* if  $i < j$ . All networks defined in this paper are built only of standard comparators. For an  $N$ -register network  $A = S_1, S_2, \dots, S_d$ , where  $S_1, S_2, \dots, S_d$  denote stages, and for an integer  $j \in \{1, \dots, N\}$ , we will use the following notations:

$$\begin{aligned} fst(j, A) &= \min \{1 \leq i \leq d : j \in \text{regs}(S_i)\} \\ lst(j, A) &= \max \{1 \leq i \leq d : j \in \text{regs}(S_i)\} \\ delay(A) &= \max_{j \in \{1, \dots, N\}} \{lst(j, A) - fst(j, A) + 1\} \end{aligned}$$

where  $\text{regs}(\{[i_1 : j_1], \dots, [i_r : j_r]\})$  denotes the set  $\{i_1, j_1, \dots, i_r, j_r\}$ .

Let us define formally the new family of merging networks. For each  $k \geq p \geq 4$  we would like to transform the network  $CW_k$  into a new network  $P_k^p$ .

**Definition 1.** Let  $n_k = 2^{k-1} - 1$  be one less than the half of the number of registers in  $CW_k$ ,  $b_k^p = 2 \lceil \frac{k-2}{p-2} \rceil$  and  $D_k^p = k - 1 + \frac{b_k^p}{2}$ . The number of registers of  $P_k^p$  is defined to be  $N_k^p + 2$ , where  $N_k^p = n_k \cdot b_k^p$ . The stages

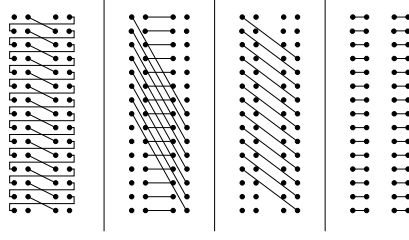


Figure 3: The  $M_5^4$  network that is 4-periodic.

of  $P_k^p = S_{k,1}^p \cup \{[0 : 1], [N_k : N_k + 1]\}$ ,  $S_{k,2}^p, \dots, S_{k,D_k^p}^p$  are defined by the following equations:

$$\begin{aligned} S_{k,1}^p &= \{[b_k^p i : b_k^p i + 1] : i = 1, \dots, n_k - 1\} \\ S_{k,j+s}^p &= \left\{ [b_k^p i + j : b_k^p (i + 2^{k-s-1} - 1) + (b_k^p - j + 1)] : i = 0, \dots, n_k - 2^{k-s-1} \right\} \\ &\quad \text{where } 1 \leq j \leq \frac{b_k^p}{2} \text{ and } (p-2)(j-1) < s \leq \min((p-2)j, k-1); \\ S_{k,(p-1)j+1}^p &= \{[b_k^p i + j : b_k^p i + j + 1], [b_k^p i + (b_k^p - j) : b_k^p i + (b_k^p - j + 1)] : i = 0, \dots, n_k - 1\}, \\ &\quad \text{where } 1 \leq j \leq \frac{k-2}{p-2}. \end{aligned}$$

The networks  $P_5^4$  and  $P_6^4$  are depicted in Figures 2 and 7, respectively.

**Fact 1.**  $\text{delay}(P_k^p) = p$  for any  $k \geq p \geq 4$ . □

Let  $A = S_1, S_2, \dots, S_d$  and  $A' = S'_1, S'_2, \dots, S'_{d'}$  be  $N$ -input comparator networks such that for each  $i$ ,  $1 \leq i \leq \min(d, d')$ ,  $\text{regs}(S_i) \cap \text{regs}(S'_i) = \emptyset$ . Then  $A \cup A'$  is defined to be a network with stages  $(S_1 \cup S'_1), (S_2 \cup S'_2), \dots, (S_{\max(d, d')} \cup S'_{\max(d, d')})$ , where empty stages are added at the end of the network of smaller depth.

For any comparator network  $A = S_1, \dots, S_d$  and  $D = \text{delay}(A)$ , let us define a network  $B = T_1, \dots, T_D$  to be a *compact form* of  $A$ , where  $T_q = \bigcup \{S_{q+pD} : 0 \leq p \leq (d-q)/D\}$ ,  $1 \leq q \leq D$ . Observe that  $B$  is correctly defined due to the delay of  $A$ . Moreover,  $\text{depth}(B) = \text{delay}(B) = \text{delay}(A)$ .

**Definition 2.** For  $k \geq p \geq 4$  let  $M_k^p$  denote the compact form of  $P_k^p$  with the first and the last registers deleted. That is, the network  $M_k^p = T_1^{p,k}, \dots, T_p^{p,k}$  is using the set of registers numbered  $\{1, 2, \dots, N_k^p\}$ , where  $N_k^p = n_k \cdot b_k^p$ ,  $n_k = 2^{k-1} - 1$ ,  $b_k^p = 2^{\lceil \frac{k-2}{p-2} \rceil}$ , and for  $j = 1, \dots, p$  the stage  $T_j^{p,k}$  is defined as  $\bigcup \{S_{k,j+pi}^p : 0 \leq i \leq \frac{D_k^p - j}{p}\}$ , where  $D_k^p = k - 1 + \frac{b_k^p}{2}$ .

It is not necessary to delete the first and the last registers of  $P_k^p$  but this will simplify proofs a little bit in the next section. The network  $M_5^4$  is given in Fig. 3.

**Theorem 1.** For any  $p \geq 4$  there exists a family of  $p$ -periodic comparator networks  $M_k^p$ ,  $k \geq p$ , such that each  $M_k^p$  is a  $p$ -periodic,  $(b_k^p - 1)$ -pass merger of two sorted sequences given in odd and even registers, respectively. The running time of  $M_k^p$  is  $p(b_k - 1) \leq \frac{2p}{p-2}k + p \frac{p-8}{p-2} \leq \frac{2p}{p-2} \log N_k^p + p \frac{p-8}{p-2}$ , where  $b_k^p = 2^{\lceil \frac{k-2}{p-2} \rceil}$  and  $N_k^p = (2^{k-1} - 1) \cdot b_k^p$  is the number of registers in  $M_k^p$ .

This is the main theorem of the paper. The rest of paper is devoted to its proof, which is based on the general observation that  $M_k^p$  merges  $\lceil \frac{k-2}{p-2} \rceil$  pairs of sorted subsequences, one after another, in pipeline fashion. Details are given in the next section.

### 3 Proof of Theorem 1

The first observation we would like to make is that we can consider inputs consisting only of 0's and 1's. The well-known Zero-One Principle states that any comparator network that sorts 0-1 input sequences

correctly sorts also arbitrary input sequences [1]. In the similar way, one can prove that the same property holds also for merging:

**Proposition 2.** *If a comparator network merges any two 0-1 sorted sequences, then it correctly merges any two sorted sequences.*  $\square$

It follows that we can analyze computations of the network  $M_k^p$ ,  $k \geq p \geq 4$ , by describing each state of registers as a 0-1 sequence  $\bar{x} = (x_1, \dots, x_{N_k^p})$ , where  $x_i$  represents the content of register  $i$ . If  $\bar{x}$  is an input sequence for  $b_k^p - 1$  passes of  $M_k$ , then by  $\bar{x}^{(i)}$  we denote the content of registers after  $i$  passes of  $M_k^p$ ,  $i = 0, \dots, b_k^p - 1$ , that is,  $\bar{x}^{(0)} = \bar{x}$  and  $\bar{x}^{(i+1)} = M_k^p(\bar{x}^{(i)})$ . Since  $M_k^p$  consists of  $p$  stages  $T_1^{p,k}, \dots, T_p^{p,k}$ , we extend the notation to describe the output of each stage:  $\bar{x}^{(i,0)} = \bar{x}^{(i)}$  and  $\bar{x}^{(i,j)} = T_j^{p,k}(\bar{x}^{(i,j-1)})$ , for  $j = 1, \dots, p$ . For other values of  $j$  we assume that  $\bar{x}^{(i,j)} = \bar{x}^{(i+j \div p, j \bmod p)}$ . We will use this superscript notation for other equivalent representations of sequence  $\bar{x}$ .

Now let us fix some technical notations and definitions. A 0-1 sequence can be represented as a word over  $\Sigma = \{0, 1\}$ . A non-decreasing (also called *sorted*) 0-1 sequence has a form of  $0^*1^*$  and can be equivalently represented by the number of ones (or zeros) in it. For any  $x \in \Sigma^*$  let  $\text{ones}(x)$  denote the number of 1 in  $x$ . If  $x \in \Sigma^n$  then  $x_i$ ,  $1 \leq i \leq n$ , denotes the  $i$ -th letter of  $x$ . Generally, for  $A = \{i_1, \dots, i_m\}$ ,  $1 \leq i_1 < \dots < i_m \leq n$ , let  $x_A$  denotes the word  $x_{i_1} \dots x_{i_m}$ . We say that a sequence  $\bar{x} = (x_1, \dots, x_{N_k})$  is *2-sorted* if both  $(x_1, x_3, \dots, x_{N_k-1})$  and  $(x_2, x_4, \dots, x_{N_k})$  are sorted.

The roadmap of the proof in the next three subsections is as follows:

1. In Subsection 3.1 we reduce the analysis of periodic applications of our  $p$  stages to a 0-1 input to an analysis of periodic applications of  $p$  quite simple functions to a short sequence of integers representing the numbers of ones in columns.
2. In Subsection 3.2 we start the analysis of computations on sequences with, so called, *balanced* sequences. A sequence  $(c_1, c_2, \dots, c_{2n})$  is called balanced if  $c_1 + c_{2n} = c_2 + c_{2n-1} = \dots = c_n + c_{n+1}$ . Being balanced is preserved by the simple functions.
3. In subsection 3.3 we use balanced sequences as upper and lower bounds on unbalanced sequences. The analysed functions are monotone.

### 3.1 Reduction to Analysis of Columns

For any  $k \geq p \geq 4$  let  $n_k = 2^{k-1} - 1$ ,  $b_k^p = 2^{\lceil \frac{k-2}{p-2} \rceil}$  (thus  $N_k^p = n_k \cdot b_k^p$ ) and  $D_k^p = k - 1 + \frac{b_k^p}{2}$ . The set of registers  $\text{Reg}_k^p = \{1, \dots, N_k^p\}$  can be analysed as an  $n_k \times b_k^p$  matrix with  $C_j^{p,k} = \{j + ib_k^p : 0 \leq i < n_k\}$ ,  $j = 1, \dots, b_k^p$ , as columns. A content of all registers in the matrix, that is  $x \in \Sigma^{N_k}$ , can be equivalently represented by the sequence of contents of registers in  $C_1^{p,k}, C_2^{p,k}, \dots, C_{b_k^p}^{p,k}$ , that is  $(x_{C_1^{p,k}}, \dots, x_{C_{b_k^p}^{p,k}})$ . Since  $b_k^p$  is an even number, the following fact is obviously true.

**Fact 2.** *If  $x \in \Sigma^{N_k^p}$  is 2-sorted then each  $x_{C_j^{p,k}}$ ,  $j = 1, \dots, b_k^p$ , is sorted.*  $\square$

That is, the columns are sorted at the beginning of a computation of  $b_k^p - 1$  passes of  $M_k^p$ . The first lemma we would like to prove is that columns remain sorted after each stage of the computation. We start with a following technical fact:

**Fact 3.** *Let  $A = \{a_1, \dots, a_n\}$  and  $B = \{b_1, \dots, b_n\}$  be subsets of  $\{1, \dots, N_k^p\}$  such that  $a_1 < b_1 < a_2 < b_2 < \dots < a_n < b_n$ . Let  $h \geq 0$  and  $S_{A,B,h} = \{[a_i : b_{i+h}] : 1 \leq i \leq n - h\}$ . Then for any  $x \in \Sigma^{N_k^p}$  such that  $x_A$  and  $x_B$  are sorted, the output  $y = S_{A,B,h}(x)$  has the following properties:*

- (i)  $y_A$  and  $y_B$  are sorted.
- (ii) Let  $m_1 = \text{ones}(x_A)$  and  $m_2 = \text{ones}(x_B)$ . Then  $\text{ones}(y_A) = \min(m_1, m_2 + h)$  and  $\text{ones}(y_B) = \max(m_1 - h, m_2)$ .

*Proof.* To prove (i) we show only that  $y_{a_i} \leq y_{a_{i+1}}$  for  $i = 1, \dots, n-1$ . If  $1 \leq i < n-h$  then  $y_{a_i} = \min(x_{a_i}, x_{b_{i+h}}) \leq \min(x_{a_{i+1}}, x_{b_{i+h+1}}) = y_{a_{i+1}}$  since  $\min$  is a non-decreasing function and both  $x_A$  and  $x_B$  are sorted. If  $i = n-h$  then  $y_{a_i} = \min(x_{a_i}, x_{b_{i+h}}) \leq x_{a_{i+1}} = y_{a_{i+1}}$ . For  $i > n-h$  we have  $y_{a_i} = x_{a_i} \leq x_{a_{i+1}} = y_{a_{i+1}}$ .

To prove (ii) let  $m'_1 = \min(m_1, m_2 + h)$  and  $m'_2 = \max(m_1 - h, m_2)$ . We consider two cases. If  $m_1 \leq m_2 + h$  then  $m_1 - h \leq m_2$  and we get  $m'_1 = m_1$  and  $m'_2 = m_2$ . In this case no comparator from  $S_{A,B,h}$  exchanges 0 with 1. To see this assume a.c. that a comparator  $[a_i : b_{i+h}]$  exchanges  $x_{a_i} = 1$  with  $x_{b_{i+h}} = 0$ . Then  $i > n - m_1$  and  $i + h \leq n - m_2$  hold because of the definitions of  $m_1$  and  $m_2$ . It follows that  $n - m_1 < n - m_2 - h$ , thus  $m_1 - h > m_2$  — a contradiction. If  $m_1 > m_2 + h$  then  $m'_1 = m_2 + h$  and  $m'_2 = m_1 - h$ . In this case let us observe that a comparator  $[a_i : b_{i+h}]$  exchanges  $x_{a_i} = 1$  with  $x_{b_{i+h}} = 0$  if and only if  $m_2 + h \leq n - i < m_1$ . Therefore  $\text{ones}(y_A) = m_1 - (m_1 - m_2 - h) = m_2 + h$  and  $\text{ones}(y_B) = m_2 + (m_1 - m_2 - h) = m_1 - h$ .  $\square$   $\square$

Since now on we continue the proof for a fixed value  $p \geq 4$  and omit  $p$  in superscripts/subscripts of our denotations, for example, we write  $M_k$  instead of  $M_k^p$ .

According to the definition of  $M_k$ , it consists of stages  $T_1^k, \dots, T_p^k$ , where  $T_i^k = \bigcup \{S_{k,i+pj} : 0 \leq j \leq \frac{D_k-i}{p}\}$  (sets  $S_{k,j}$  are defined in Def. 1). Using the notation from Fact 3, the following fact is an easy consequence of Definition 1.

**Fact 4.** Let  $L_i = C_i^k$  and  $R_i = C_{b_k-i+1}^k$  denote the corresponding left and the right columns of registers, and  $h_i = 2^{k-i-1} - 1$ ,  $i = 1, \dots, \frac{b_k}{2}$ . Then

- (i)  $\text{regs}(S_{k,1}) \subseteq L_1 \cup R_1$  and  $S_{k,1} = S_{R_1 - \{N_k\}, L_1 - \{1\}, 0}$ ;
- (ii)  $\text{regs}(S_{k,j+s}) \subseteq L_j \cup R_j$  and  $S_{k,j+s} = S_{L_j, R_j, h_s}$ , for  $j = 1, \dots, \frac{b_k}{2}$  and  $(p-2)(j-1) < s \leq \min((p-2)j, k-1)$ ;
- (iii)  $\text{regs}(S_{k,(p-1)j+1}) \subseteq L_j \cup L_{j+1} \cup R_{j+1} \cup R_j$  and  $S_{(p-1)j+1} = S_{L_j, L_{j+1}, 0} \cup S_{R_{j+1}, R_j, 0}$ , for  $j = 1, \dots, \frac{b_k}{2} - 1$ ;
- (iv)  $\text{regs}(S_{k,D_k}) \subseteq L_{b_k/2} \cup R_{b_k/2}$  and  $S_{k,D_k} = S_{L_{b_k/2}, R_{b_k/2}, 0}$ ;
- (v) if  $(L_j \cup R_j) \cap \text{regs}(S_{k,i}) \neq \emptyset$  then  $(p-1)(j-1) + 1 \leq i \leq \min((p-1)j+1, D_k)$ , for any  $j = 1, \dots, \frac{b_k}{2}$ .  $\square$

**Lemma 1.** If the initial content of registers is a 2-sorted 0-1 sequence  $x$  then after each stage of multi-pass computation of  $M_k = T_1^k, \dots, T_p^k$  the content of each column  $C_j^k$ ,  $j = 1, \dots, b_k$ , is sorted, that is, each  $(x^{(s,i)})_{C_j^k}$  is of the form  $0^*1^*$ ,  $s = 0, \dots, i = 1, \dots, p$ .

*Proof.* By induction it suffices to prove that for each sequence  $y \in \Sigma^{N_k}$  with sorted columns  $C_j^k$ ,  $j = 1, \dots, b_k$ , the outputs  $z_i = T_i^k(y)$ ,  $i = 1, \dots, p$  have also the columns sorted. Since each  $T_i^k$ , as a mapping, is a composition of mapping  $S_{k,i+pj}$ ,  $0 \leq j \leq \lfloor \frac{D_k-i}{p} \rfloor$ , each of which, due to Facts 3 and 4, transforms sorted columns into sorted columns, the lemma follows.  $\square$   $\square$

From now on, instead of looking at 0-1 sequences with sorted columns, we will analyse the computations of  $M_k$  on sequences of integers  $\bar{c} = (c_1, \dots, c_{b_k})$ , where  $c_t$ ,  $t = 1, \dots, b_k$ , denote the number of ones in a sorted column  $C_t^k$ . Transformations of 0-1 sequences defined by sets  $S_{k,j}$ ,  $j = 1, \dots, D_k$  will be represented by the following mappings:

**Definition 3.** Let  $k \geq p$ ,  $h_i = 2^{k-i-1} - 1$  for  $i = 1, \dots, k-1$  and  $b_k = 2^{\lceil \frac{k-2}{p} \rceil}$ . For  $j = 1, \dots, \frac{b_k}{2}$  and  $s = 1, \dots, k-1$  the functions  $\text{dec}_{j,s}^k$ ,  $\text{mov}_j^k$  and  $\text{cyc}^k$  over sequences of  $b_k$  reals are defined as follows. Let

$\bar{c} = (c_1, \dots, c_{b_k})$  and  $t \in \{1, \dots, b_k\}$ .

$$\begin{aligned} (dec_{j,s}^k(\bar{c}))_t &= \begin{cases} \min(c_j, c_{b_k-j+1} + h_s) & \text{if } t = j \\ \max(c_j - h_s, c_{b_k-j+1}) & \text{if } t = b_k - j + 1 \\ c_t & \text{otherwise} \end{cases} \\ (mov_j^k(\bar{c}))_t &= \begin{cases} \min(c_t, c_{t+1}) & \text{if } t = j \text{ or } t = b_k - j \\ \max(c_{t-1}, c_t) & \text{if } t = j + 1 \text{ or } t = b_k - j + 1 \\ c_t & \text{otherwise} \end{cases} \\ (cyc^k(\bar{c}))_t &= \begin{cases} \max(c_1, c_{b_k} - 1) & \text{if } t = 1 \\ \min(c_1 + 1, c_{b_k}) & \text{if } t = b_k \\ c_t & \text{otherwise} \end{cases} \end{aligned}$$

**Fact 5.** Let  $x \in \Sigma^{N_k}$  be a 0-1 sequence with sorted columns  $C_1^k, \dots, C_{b_k}^k$ , let  $c_i = \text{ones}(x_{C_i^k})$  and  $\bar{c} = (c_1, \dots, c_{b_k})$ . Let  $y_j = S_{k,j}(x)$ ,  $d_{j,i} = \text{ones}((y_j)_{C_i^k})$  and  $\bar{d}_j = (d_{j,1}, \dots, d_{j,b_k})$ , where  $i = 1, \dots, b_k$  and  $j = 1, \dots, D_k$ . Then

- (i)  $\bar{d}_1 = cyc^k(\bar{c})$
- (ii)  $\overline{d_{j+s}} = dec_{j,s}^k(\bar{c})$ , for any  $j = 1, \dots, \frac{b_k}{2}$  and  $(p-2)(j-1) < s \leq \min((p-2)j, k-1)$
- (iii)  $\overline{d_{(p-1)j+1}} = mov_j^k(\bar{c})$ , for any  $1 \leq j \leq \frac{k-2}{p-2}$

*Proof.* Generally, the fact follows from Fact 4 and the part (ii) of Fact 3. We prove only its parts (i) and (ii). Part (iii) can be proved in a similar way.

(i) Observe that  $y_1 = S_{k,1}(x) = S_{R_1 - \{N_k\}, L_1 - \{1\}, 0}(x)$  due to Fact 4 (ii). It follows that only the content of columns  $L_1 = C_1^k$  and  $R_1 = C_{b_k}^k$  can change, but they remain sorted (according to Lemma 1). Using Fact 3 (ii) we have:  $m_1 = \text{ones}(x_{R_1 - \{N_k\}}) = c_{b_k} - x_{N_k}$ ,  $m_2 = \text{ones}(x_{L_1 - \{1\}}) = c_1 - x_1$  and

$$d_{1,1} = \max(m_1, m_2) + x_1 = \max(c_{b_k} - x_{N_k} + x_1, c_1),$$

$$d_{1,b_k} = \min(m_1, m_2) + x_{N_k} = \min(c_{b_k}, c_1 + x_{N_k} - x_1).$$

Now let us consider the following three cases of values  $x_1$  and  $x_{N_k}$ :

**Case  $x_1 = 0$  and  $x_{N_k} = 1$ .** Then  $d_{1,1} = \max(c_{b_k} - 1, c_1) = cyc^k(\bar{c})_1$  and  $d_{1,b_k} = \min(c_{b_k}, c_1 + 1) = cyc^k(\bar{c})_{b_k}$ .

**Case  $x_1 = 1$ .** Then  $c_1 = n_k$ ,  $c_{b_k} \leq n_k$  and  $c_{b_k} - x_{N_k} \leq n_k - 1$ . In this case:  $d_{1,1} = \max(n_k, c_{b_k} - x_{N_k} + 1) = n_k = \max(c_1, c_{b_k} - 1) = cyc^k(\bar{c})_1$  and  $d_{1,b_k} = \min(n_k - 1 + x_{N_k}, c_{b_k}) = c_{b_k} = \min(c_1 + 1, c_{b_k}) = cyc^k(\bar{c})_{b_k}$ .

**Case  $x_{N_k} = 0$ .** Then  $c_{b_k} = 0$  and  $c_1 - x_1 \geq 0$ . In this case:  $d_{1,1} = \max(c_1, x_1) = c_1 = \max(c_1, c_{b_k} - 1) = cyc^k(\bar{c})_1$  and  $d_{1,b_k} = \min(c_1 - x_1, c_{b_k}) = c_{b_k} = \min(c_1 + 1, c_{b_k}) = cyc^k(\bar{c})_{b_k}$ .

(ii) We fix any  $j \in \{1, \dots, \frac{b_k}{2}\}$  and  $(p-2)(j-1) < s \leq \min((p-2)j, k-1)$  and observe that  $y_{j+s} = S_{k,j+s}(x) = S_{L_j, R_j, h_s}(x)$  due to Fact 4 (ii). It follows that only the content of columns  $L_j = C_j^k$  and  $R_j = C_{b_k-j+1}^k$  can change, but they remain sorted (according to Lemma 1). Using Fact 3 (ii) we have:

$$d_{j+s,j} = \text{ones}((y_{j+s})_{L_j}) = \min(c_j, c_{b_k-j+1} + h_s) = (dec_{j,s}^k(\bar{c}))_j,$$

$$d_{j+s,b_k-j+1} = \text{ones}((y_{j+s})_{R_j}) = \max(c_j - h_s, c_{b_k-j+1}) = (dec_{j,s}^k(\bar{c}))_{b_k-j+1}.$$

□

□

**Definition 4.** Let  $k \geq p$ . For  $x = 1, \dots, k$  let  $MV_x^k = \{mov_j^k : 1 \leq j \leq \frac{k-2}{p-2} \text{ and } x+j \equiv 1 \pmod{p}\}$  and  $DC_x^k = \{dec_{j,s}^k : 1 \leq j \leq \frac{b_k}{2} \text{ and } (x+j) \bmod p \notin \{1, 2\} \text{ and } s = (p-2)(j-1) - 1 + (x+j-1) \bmod p \leq k-1\}$ . Let  $Q_1^k, \dots, Q_p^k$  denote the following sets of functions.

$$Q_1^k = \{cyc^k\} \cup MV_1^k \cup DC_1^k$$

$$Q_i^k = MV_i^k \cup DC_i^k \quad \text{for } 2 \leq i \leq p.$$

Let us observe that each function in  $Q_i^k, i = 1, \dots, p$ , can modify only a few positions in a given sequence of numbers. Moreover, different functions in  $Q_i^k$  modify disjoint sets of positions. For a function  $f : R^m \mapsto R^m$  let us define

$$\text{args}(f) = \{i \in \{1, \dots, m\} : \exists \bar{c} \in R^m (f(\bar{c}))_i \neq (\bar{c})_i\}$$

The following facts formalize our observations.

**Fact 6.** Let  $k \geq p$ . Then  $\text{args}(\text{cyc}^k) = \{1, b_k\}$ ,  $\text{args}(\text{dec}_{j,s}^k) = \{j, b_k - j + 1\}$ ,  $\text{args}(\text{mov}_j^k) = \{j, j + 1, b_k - j, b_k - j + 1\}$ , where  $j = 1, \dots, \frac{b_k}{2}$ .

**Fact 7.** For each pair of functions  $f, g \in Q_i^k, f \neq g, i = 1, \dots, p$ , we have

(i)  $\text{args}(f) \cap \text{args}(g) = \emptyset$ ;

(ii) for any  $\bar{c} = (c_1, \dots, c_{b_k})$  and  $j \in \{1, \dots, b_k\}$

$$(f(g(\bar{c})))_j = \begin{cases} (f(\bar{c}))_j & \text{if } j \in \text{args}(f) \\ (g(\bar{c}))_j & \text{if } j \in \text{args}(g) \\ c_j & \text{otherwise} \end{cases}$$

*Proof.* (i) Assume a.c. that there exist  $1 \leq x \leq p, f, g \in Q_x^k$  and  $1 \leq j \leq b_k/2$  such that  $f \neq g$  and  $j \in \text{args}(f) \cap \text{args}(g)$ . Obviously, functions  $f$  and  $g$  cannot be both in  $MV_x^k$  or  $DC_x^k$ . Assume that  $f \in MV_x^k \cup \{\text{cyc}^k\}$  and  $g \in DC_x^k$ . Then  $(x + j) \bmod p \in \{1, 2\}$  from the first assumption and  $(x + j) \bmod p \notin \{1, 2\}$  from the second one - a contradiction.  $\square$

**Corollary 1.** Each set  $Q_i^k, i = 1, \dots, p$ , uniquely determines a mapping, in which functions from  $Q_i^k$  can be apply in any order. Moreover, if  $f \in Q_i^k, \bar{c} \in R^{b_k}$  and  $j \in \text{args}(f)$  then  $(Q_i^k(\bar{c}))_j = (f(\bar{c}))_j$ .  $\square$

We would like to prove that the result of applying each  $Q_i^k, i = 1, \dots, p$ , to a sequence  $\bar{c} = (c_1, \dots, c_{b_k})$  of numbers of ones in columns  $C_1^k, \dots, C_{b_k}^k$  is equivalent to applying the set of comparators  $T_i^k$  to the content of registers, if each column is sorted.

**Lemma 2.** Let  $x \in \Sigma^{N_k}$  be a 0-1 sequence with sorted columns  $C_1^k, \dots, C_{b_k}^k$ , let  $c_i = \text{ones}(x_{C_i^k})$  and  $\bar{c} = (c_1, \dots, c_{b_k})$ . Let  $y_j = T_j^k(x)$ ,  $d_{j,i} = \text{ones}((y_j)_{C_i^k})$  and  $\bar{d}_j = (d_{j,1}, \dots, d_{j,b_k})$ , where  $i = 1, \dots, b_k$  and  $j = 1, \dots, p$ . Then  $Q_j^k(\bar{c}) = \bar{d}_j$ .

*Proof.* Recall that  $T_j^k = \bigcup \{S_{k,j+pi} : 0 \leq i \leq \frac{D_k-j}{p}\}$ . For a set of comparators  $S$  let us define

$$\text{cols}(S) = \left\{ i \in \{1, \dots, b_k\} : \text{regs}(S) \cap C_i^k \neq \emptyset \right\}.$$

From Fact 4(i-iv) it follows that  $\text{cols}(S_{k,1}) = \{1, b_k\}$  and for  $j = 1, \dots, \frac{b_k}{2}$   $\text{cols}(S_{k,j+s}) = \{j, b_k - j + 1\}$  and  $\text{cols}(S_{k,(p-1)j+1}) = \{j, j + 1, b_k - j, b_k - j + 1\}$ . From Fact 4(v) we get that  $\text{cols}(S_{k,j+pi}) \cap \text{cols}(S_{k,j+pi'}) = \emptyset$  if  $i \neq i'$ . Thus we can observe a 1-1 correspondence between a function  $f$  in  $Q_j^k$  and a set of comparators  $S_{k,j+pi} \subseteq T_j^k$  such that  $\text{args}(f) = \text{cols}(S_{k,j+pi})$ . Then for each  $t \in \text{args}(f)$   $(Q_j^k(\bar{c}))_t = (f(\bar{c}))_t = (\bar{d}_j)_t$ , as the consequence of Corollary 1 and Fact 5.  $\square$

**Definition 5.** We say that a sequence of numbers  $\bar{c} = (c_1, \dots, c_{2m})$  is *flat* if  $c_1 \leq c_2 \leq \dots, c_{2m} \leq c_1 + 1$ . We say that a sequence  $\bar{c}$  is *2-flat* if subsequences  $(c_1, c_3, \dots, c_{2m-1})$  and  $(c_2, c_4, \dots, c_{2m})$  are flat. We say that  $\bar{c}$  is *balanced* if  $c_i + c_{2m-i+1} = c_1 + c_{2m}$ , for  $i = 2, \dots, m$ . For a balanced sequence  $\bar{c}$  define *height*( $\bar{c}$ ) as  $c_1 + c_{2m}$ .

**Proposition 3.** Let  $k \geq p, x \in \Sigma^{N_k}, \bar{c} = (c_1, \dots, c_{b_k})$ , where  $c_i = \text{ones}(x_{C_i^k})$  ( $C_i^k$  is as usual a column in the matrix of registers),  $i = 1, \dots, b_k$ . Then

(i)  $x$  is sorted if and only if columns of  $x$  are sorted and  $\bar{c}$  is flat;



(ii)  $x$  is 2-sorted if and only if columns of  $x$  are sorted and  $\bar{c}$  is 2-flat;

Now we are ready to reduce the proof of Theorem 1 to the proof of following lemma.

**Lemma 3.** *Let  $k \geq p \geq 4$  and  $b_k = 2^{\lceil \frac{k-2}{p-2} \rceil}$ . If for each 2-flat sequence  $\bar{c} = (c_1, \dots, c_{b_k})$  of integers from  $[0, 2^{k-1} - 1]$  the result of application  $(Q_p^k \circ \dots \circ Q_1^k)^{b_k-1}$  to  $(\bar{c})$  is a flat sequence, then  $M_k$  is a  $(b_k - 1)$ -pass merger of two sorted sequences given in odd and even registers, respectively.*

*Proof.* Assume that for each 2-flat sequence  $\bar{c} = (c_1, \dots, c_{b_k})$  the result of application  $(Q_p^k \circ \dots \circ Q_1^k)^{b_k-1}$  to  $(\bar{c})$  is a flat sequence. Let  $\bar{x} \in \Sigma^{N_k}$  be a 2-sorted sequence and  $\bar{c} = (c_1, \dots, c_{b_k})$ , where  $c_i = \text{ones}(\bar{x}_{C_i^k})$  ( $C_i^k$  is as usual a column in the matrix of registers),  $i = 1, \dots, b_k$ . Then  $\bar{c}$  is 2-flat due to Proposition 3 and each  $c_i \in [0, 2^{k-1} - 1]$ , because the height of columns is  $2^{k-1} - 1$ . Recall that  $\bar{x}^{(j)} = (M_k)^j(\bar{x})$  and let  $c_{j,i} = \text{ones}(\bar{x}_{C_i^k}^{(j)})$ . Using Lemma 2 and easy induction we get that the equality  $(Q_p^k \circ \dots \circ Q_1^k)^j(\bar{c}) = (c_{j,1}, \dots, c_{j,b_k})$  is true for  $j = 1, \dots, b_k - 1$ . Since the result of  $(Q_p^k \circ \dots \circ Q_1^k)^{b_k-1}(\bar{c})$  is a flat sequence, the sequence  $\bar{x}^{(b_k-1)}$  is sorted.  $\square$

## 3.2 Analysis of Balanced Columns

Due to Lemma 3 we can analyse only the results of periodic application of the functions  $Q_1^k, \dots, Q_p^k$  to a sequence of integers representing the numbers of ones in each register column. We know also that an initial sequence is 2-flat. To simplify our analysis further, we start it with initial values restricted to be balanced 2-flat sequences. In this section we prove that after  $p(b_k - 1) - b_k/2 + 1$  such application to a balanced 2-flat sequence we get a flat output sequence (see Lemma 8). Then we observe that the functions are monotone and any 2-flat sequence can be bounded from below and above by balanced 2-flat sequences whose heights differ at most by one. Using these facts we analyse general 2-flat sequences in the next section.

**Lemma 4.** *Let  $k \geq p$  and  $\bar{c} = (c_1, \dots, c_{b_k})$  be a balanced sequence of numbers. Let  $s = \text{height}(\bar{c})$  and let  $f$  be a function from  $Q_1^k \cup \dots \cup Q_p^k$ . Then  $f(\bar{c})$  is also balanced and  $\text{height}(f(\bar{c})) = s$ .*

*Proof.* Let  $\bar{c}$  and  $s$  be as assumed in the lemma and let  $f(\bar{c}) = (d_1, \dots, d_{b_k})$ . The function  $f \in Q_1^k \cup \dots \cup Q_p^k$  can be either  $\text{cyc}^k$  or one of  $\text{mov}_{j,t}^k, \text{dec}_{j,t}^k$ , where  $j = 1, \dots, b_k/2$  and  $t = 1, \dots, k - 1$ , according to Definition 4. Each of the functions can modify only one or two pairs of positions of the form  $(i, b_k - i + 1)$  in  $\bar{c}$  (see Fact 6). The other pairs are left untouched, so the sum of their values cannot change. In case of  $\text{cyc}^k$  the modified pair is  $(1, b_k)$  and  $d_1 + d_{b_k} = \max(c_1, c_{b_k} - 1) + \min(c_1 + 1, c_{b_k}) = s$ . In case of  $\text{dec}_{j,t}^k$  the pair is  $(j, b_k - j + 1)$  and  $d_j + d_{b_k-j+1} = \min(c_j, c_{b_k-j+1} + h_t) + \max(c_j - h_t, c_{b_k-j+1}) = \min(c_j - h_t, c_{b_k-j+1}) + h_t + \max(c_j - h_t, c_{b_k-j+1}) = s$ . Finally, if  $f = \text{mov}_{j,t}^k$  then we have two pairs  $(j, b_k - j + 1)$  and  $(j + 1, b_k - j)$ . Then  $d_j + d_{b_k-j+1} = \min(c_j, c_{j+1}) + \max(c_{b_k-j}, c_{b_k-j+1}) = \min(c_j, c_{j+1}) + \max(s - c_{j+1}, s - c_j) = s$ . In case of the other pair  $d_{j+1} + d_{b_k-j} = \max(c_j, c_{j+1}) + \min(c_{b_k-j}, c_{b_k-j+1}) = \max(c_j, c_{j+1}) + \min(s - c_{j+1}, s - c_j) = s$ .  $\square$

It follows from Lemma 4 that if we start the periodic application of the functions  $Q_1^k, \dots, Q_p^k$  to a balanced 2-flat initial sequence then it remains balanced after each function application and its height will not changed. Therefore, we can trace only the values in the first half of generated sequences. If needed, a value in the second half can be computed from the height and the corresponding value in the first half. To get a better view on the structure of generated sequences, we subtract half of the height from each element of the initial sequence and proceed with such modified sequences to the end. At the end the subtracted value is added to each element of the final sequence. The following fact justifies the described above procedure.

**Fact 8.** *Let  $f$  be a function from  $Q_1^k \cup \dots \cup Q_p^k$ . Then  $f$  is monotone and for each  $t \in R$  and  $(c_1, \dots, c_{b_k}) \in R^{b_k}$  the following equation is true*

$$f(c_1 - t, \dots, c_{b_k} - t) = f(c_1, \dots, c_{b_k}) - (t, \dots, t) .$$

*Proof.* The fact follows from the similar properties of min and max functions: they are monotone and the equations:  $\min(x - t, y - t) = \min(x, y) - t$  and  $\max(x - t, y - t) = \max(x, y) - t$  are obviously true. Each  $f$  in  $Q_1^k \cup \dots \cup Q_p^k$  is defined with the help of these simple functions, thus  $f$  inherits the properties.  $\square$

**Corollary 2.** Let  $f = f_l \circ f_{l-1} \circ \dots \circ f_1$ , where each  $f_i$  is from  $\{Q_1^k, \dots, Q_p^k\}$ ,  $1 \leq i \leq l$ . Then  $f$  is monotone and for any  $t \in R$  and  $(c_1, \dots, c_{b_k}) \in R^{b_k}$

$$f(c_1 - t, \dots, c_{b_k} - t) = f(c_1, \dots, c_{b_k}) - (t, \dots, t) .$$

**Definition 6.** Let  $\bar{c} = (c_1, \dots, c_{b_k}) \in R^{b_k}$  be a balanced sequence and  $s = \text{height}(\bar{c})$ . We call  $(c_1 - \frac{s}{2}, c_2 - \frac{s}{2}, \dots, c_{b_k/2} - \frac{s}{2}) \in R^{b_k/2}$  the reduced sequence of  $\bar{c}$  and denote it by  $\text{reduce}(\bar{c})$ . For a sequence  $\bar{d} = (d_1, \dots, d_{b_k/2}) \in R^{b_k/2}$  we define  $s$ -extended sequence  $\text{ext}(\bar{d}, s)$  as

$$(d_1 + \frac{s}{2}, d_2 + \frac{s}{2}, \dots, d_{b_k/2} + \frac{s}{2}, \frac{s}{2} - d_{b_k/2}, \frac{s}{2} - d_{b_k/2-1}, \dots, \frac{s}{2} - d_1) .$$

For any  $t \in R$  and a function  $f : R^{b_k} \mapsto R^{b_k}$  that maps balanced sequences to balanced ones and preserves heights let  $\text{reduce}(f, t)$  denote a function on  $R^{b_k/2}$  such that for any  $\bar{d} \in R^{b_k/2}$

$$(\text{reduce}(f, t))(\bar{d}) = \text{reduce}(f(\text{ext}(\bar{d}, t)))$$

Observe that for a balanced sequence  $\bar{c}$  with height  $s$  the sequence  $\text{ext}(\text{reduce}(\bar{c}), s)$  is equal to  $\bar{c}$ . Moreover, for any  $t \in R$  and a sequence  $\bar{d} \in R^{b_k/2}$  the sequence  $\text{ext}(\bar{d}, t)$  is balanced and its height is  $t$ , thus  $\text{reduce}(\text{ext}(\bar{d}, t)) = \bar{d}$ . Note also that functions  $Q_1^k, \dots, Q_p^k$  preserve the property of being balanced and the sequence height (see Lemma 4), so we can analyse a periodic application of their reduced forms to a reduced balanced 2-flat input.

**Fact 9.** Let  $f = f_l \circ f_{l-1} \circ \dots \circ f_1$ , where  $f_i \in \{Q_1^k, \dots, Q_p^k\}$ ,  $1 \leq i \leq l$ . Let  $\bar{c} \in R^{b_k}$  be balanced and  $s = \text{height}(\bar{c})$ . Let  $\hat{f}_i = \text{reduce}(f_i, s)$ ,  $1 \leq i \leq l$ , and  $\hat{f} = \hat{f}_l \circ \hat{f}_{l-1} \circ \dots \circ \hat{f}_1$ . Then  $f(\bar{c}) = \text{ext}((\hat{f})(\text{reduce}(\bar{c})), s)$ .

**Definition 7.** Define  $\text{MinMax}(x, y)$  to be  $(\min(x, y), \max(x, y))$ ,  $\text{Min}(x) = \min(x, -x)$  and  $\text{Cyc}(x) = \max(x, -x - 1)$ . Let  $\text{Dec}_i(x) = \min(x, -x + H_i)$ , where  $H_i = 2^i - 1, i = 1, \dots$

**Fact 10.** Let  $k \geq p$ . For each  $f \in Q_1^k \cup \dots \cup Q_p^k$  and  $t \geq 0$  the function  $\text{reduce}(f, t)$  does not depend on  $t$  and for any sequence  $\bar{d} \in R^{b_k/2}$  and an index  $u$ ,  $1 \leq u \leq \frac{b_k}{2}$  the following equations are true:

- (i)  $(\text{reduce}(\text{cyc}^k, t)(\bar{d}))_u = \text{if } u = 1 \text{ then } \text{Cyc}(d_1) \text{ else } d_u;$
- (ii)  $(\text{reduce}(\text{dec}_{j,s}^k, t)(\bar{d}))_u = \text{if } u = j \text{ then } \text{Dec}_{k-s-1}(d_j) \text{ else } d_u;$
- (iii)  $(\text{reduce}(\text{mov}_{j,t}^k, t)(\bar{d}))_u = \text{if } u \in \{j, j+1\} \text{ then } (\text{MinMax}(d_j, d_{j+1}))_{u-j+1} \text{ else } d_u, \text{ for } j < b_k/2;$
- (iv)  $(\text{reduce}(\text{mov}_{b_k/2,t}^k, t)(\bar{d}))_u = \text{if } u = b_k/2 \text{ then } \text{Min}(d_{b_k/2}) \text{ else } d_u.$

*Proof.* By Lemma 4 the considered functions preserve the height of sequences and their property of being balanced, thus we can use their reduced forms and  $(\text{reduce}(f, t))(\bar{d}) = \text{reduce}(f(\text{ext}(\bar{d}, t)))$ . If  $u \notin \text{args}(f)$  then  $(\text{reduce}(f(\text{ext}(\bar{d}, t))))_u = d_u$  according to Def. 6. If  $u \in \text{args}(f)$  then we have to consider the following cases.

**Case**  $f = \text{cyc}^k$ . Then  $u$  must be equal to 1 and  $(\text{reduce}(\text{cyc}^k(\text{ext}(\bar{d}, t))))_1 = \max(d_1 + \frac{t}{2}, \frac{t}{2} - d_1 - 1) - \frac{t}{2} = \max(d_1, -d_1 - 1) = \text{Cyc}(d_1)$ .

**Case**  $f = \text{dec}_{j,s}^k$ . Then  $u$  must be equal to  $j$  and  $(\text{reduce}(\text{dec}_{j,s}^k(\text{ext}(\bar{d}, t))))_j = \min(d_j + \frac{t}{2}, \frac{t}{2} - d_j + h_s) - \frac{t}{2} = \min(d_j, -d_j + h_s) = \text{Dec}_{k-s-1}(d_j)$ , because  $h_s = 2^{k-s-1} - 1 = H_{k-s-1}$ .

**Case**  $f = \text{mov}_{j,t}^k$  and  $j < \frac{b_k}{2}$ . Then  $u \in \{j, j+1\}$ . For  $u = j$ ,  $(\text{reduce}(\text{mov}_{j,t}^k(\text{ext}(\bar{d}, t))))_j = \min(d_j + \frac{t}{2}, d_{j+1} + \frac{t}{2}) - \frac{t}{2} = \min(d_j, d_{j+1}) = (\text{MinMax}(d_j, d_{j+1}))_1$ . For  $u = j+1$  the proof is similar.

**Case**  $f = \text{mov}_{b_k/2,t}^k$  and  $j = \frac{b_k}{2}$ . Then  $u$  must be  $b_k/2$  and  $(\text{reduce}(\text{mov}_{b_k/2,t}^k(\text{ext}(\bar{d}, t))))_{b_k/2} = \min(d_{b_k/2} + \frac{t}{2}, \frac{t}{2} - d_{b_k/2}) - \frac{t}{2} = \min(d_{b_k/2}, -d_{b_k/2}) = \text{Min}(d_{b_k/2})$ .  $\square$   $\square$

**Definition 8.** Let  $k \geq p$  and for each  $f \in Q_1^k \cup \dots \cup Q_p^k$  let  $\hat{f}$  denote its reduced form  $\text{reduce}(f, *)$  (it does not depend on the second argument). Let  $\hat{Q}_1^k, \dots, \hat{Q}_p^k$  denote the following sets of reduced functions:  $\hat{Q}_i^k = \{\hat{f} : f \in Q_i^k\}$ , where  $i = 1, \dots, p$ .

**Lemma 5.** Let  $k \geq p$  and  $t \in R$ . Then the function  $\text{reduce}(Q_i^k, t)$  does not depend on  $t$  and  $\text{reduce}(Q_i^k, t) = \hat{Q}_i^k$ , where  $i = 1, \dots, p$ .

*Proof.* Let  $f$  be any function in  $Q_1^k \cup \dots \cup Q_p^k$  and let  $\hat{f}$  denote its reduced form. By Fact 10, we know that  $\text{args}(\hat{f}) = \text{args}(f) \cap \{1, \dots, \frac{b_k}{2}\}$ . By the definitions,  $\text{args}(Q_i^k) = \bigcup_{f \in Q_i^k} \text{args}(f)$  and  $\text{args}(\hat{Q}_i^k) = \bigcup_{f \in Q_i^k} \text{args}(\hat{f})$ . Consider now a sequence  $\bar{d} \in R^{b_k/2}$  and an index  $u$ ,  $1 \leq u \leq \frac{b_k}{2}$ . If  $u \notin \text{args}(Q_i^k)$  then  $(\text{reduce}(Q_i^k, t)(\bar{d}))_u = d_u = (\hat{Q}_i^k(\bar{d}))_u$ . Otherwise, if  $u \in \text{args}(f)$ ,  $f \in Q_i^k$ , then  $(\text{reduce}(Q_i^k, t)(\bar{d}))_u = (\text{reduce}(Q_i^k(\text{ext}(\bar{d}, t)))_u = (\text{reduce}(f(\text{ext}(\bar{d}, t))))_u = (\hat{f}(\bar{d}))_u = (\hat{Q}_i^k(\bar{d}))_u$ .  $\square$   $\square$

Instead of tracing individual values in reduced sequences after each application of a function from  $\{\hat{Q}_1^k, \dots, \hat{Q}_p^k\}$  we will trace intervals in which the values should be and observe how the lengths of intervals are decreasing during the computation. So let us now define the intervals and give a fact about computations on them.

**Definition 9.** Let  $k \geq 4$ ,  $H_i = 2^i - 1$  for  $1 \leq i \leq k - 1$ . Let  $I(0)$  denote the interval  $[-\frac{1}{2}, 0]$  and, in similar way, let  $I(i) = [-\frac{1}{2}, \frac{H_i}{2}]$ ,  $1 \leq i \leq k - 1$ ,  $I(-k) = [-\frac{H_{k-1}}{2}, 0]$  and  $I(\pm k) = [-\frac{H_{k-1}}{2}, \frac{H_{k-1}}{2}]$ . Moreover, we write  $I(w_1, w_2, \dots, w_l)$  for the Cartesian product  $I(w_1) \times I(w_2) \times \dots \times I(w_l)$ , where each  $w_i \in \{0, 1, 2, \dots, k - 1, -k, \pm k\}$ .

**Fact 11.** The following inclusions are true:

1.  $\text{Dec}_i(I(i+1)) \subseteq I(i)$  and  $\text{Dec}_i(I(w)) \subseteq I(w)$ , for  $1 \leq i \leq k - 2$  and  $w \in \{0, -k, \pm k\}$ ;
2.  $\text{Cyc}(I(-k)) \subseteq I(k - 1)$  and  $\text{Cyc}(w) \subseteq \text{Cyc}(w)$ , for  $w \in \{0, k - 1\}$ ;
3.  $\text{Min}(I(\pm k)) \subseteq I(-k)$  and  $\text{Min}(I(1)) \subseteq I(0)$ ;
4.  $\text{MinMax}(I(\pm k, -k)) \subseteq (I(-k, \pm k))$ ;
5.  $\text{MinMax}(I(i, w)) \subseteq (I(w, i))$ , for  $1 \leq i \leq k - 1$  and  $w \in \{0, -k\}$ .

*Proof.* The proof of each inclusion is a straightforward consequence of the definitions of a given function and intervals. Therefore we check only inclusions given in the first item. Let  $x \in I(i+1) = [-\frac{1}{2}, \frac{H_{i+1}}{2}]$ . If  $x \in I(i) = [-\frac{1}{2}, \frac{H_i}{2}]$ , then  $\text{Dec}_i(x) = \min(x, -x + H_i) = x$  since  $2x \leq H_i$ . Otherwise  $x$  must be in  $(\frac{H_i}{2}, \frac{H_{i+1}}{2}]$ , but then  $x > -x + H_i$  and  $\text{Dec}_i(x) = -x + H_i \in [-\frac{1}{2}, \frac{H_i}{2}]$  since  $H_{i+1} = 2H_i + 1$ .

To proof the second inclusion for  $\text{Dec}_i$  let us observe that if  $x \leq 0$  then  $\text{Dec}_i(x) = x$ . It follows that  $\text{Dec}_i(I(0)) \subseteq I(0)$  and  $\text{Dec}_i(I(-k)) \subseteq I(-k)$ . In case of  $x \in I(\pm k)$  we have to check only the positive values of  $x$ , such that  $x \geq -x + H_i$ . But then  $\text{Dec}_i(x) = -x + H_i > -x$  and both  $x, -x \in I(\pm k)$ .  $\square$   $\square$

Now we are ready to define sequences of intervals that are used to describe states of computation after each periodic application of functions  $\hat{Q}_1^k, \dots, \hat{Q}_p^k$  to a reduced sequence of numbers of ones in columns.

**Definition 10.** Let  $k \geq p$ . For  $0 \leq x \leq p$  and  $1 \leq l \leq b_k/2$  let

$$e_p^k(x, l) = \max(0, k - (p - 2)(l - 1) - (x + l - 1) \bmod p)$$

be an auxiliary function to define the following sequences of length  $b_k/2$

$$\begin{aligned} (U_x^k)_l &= \text{if } x + l \equiv 1 \pmod{p} \text{ then } -k \text{ else } \pm k, \\ (V_x^k)_l &= \text{if } x + l \equiv 1 \pmod{p} \text{ then } -k \text{ else } e_p^k(x, l), \\ (W_x^k)_l &= \text{if } x + l \equiv 1 \pmod{p} \text{ then } 0 \text{ else } e_p^k(x, l), \\ (Z^k)_l &= 0. \end{aligned}$$

Note that the elements of the defined above sequences are interval descriptors as defined in Definition 9 and we have also  $U_0^k = U_p^k$ ,  $V_0^k = V_p^k$  and  $W_0^k = W_p^k$ .

**Definition 11.** Let  $k \geq p$ . Let  $\bar{a} = (a_1, \dots, a_n)$  and  $\bar{b} = (b_1, \dots, b_n)$  be any sequences, where  $n \geq \frac{b_k}{2}$ . For  $0 \leq i \leq \frac{b_k}{2}$  let  $join_k(i, \bar{a}, \bar{b})$  denote  $(a_1, \dots, a_i, b_{i+1}, \dots, b_{b_k/2})$ .

**Definition 12.** Let  $k \geq p$ . Let  $X_i^k$  denote a state sequence after  $i$  stages and be defined as:

$$X_i^k = \begin{cases} join_k(\lceil \frac{i+1}{p-1} \rceil, V_{i \bmod p}^k, U_{i \bmod p}^k) & 1 \leq i \leq \frac{b_k}{2}(p-1) - 1 \\ join_k(\frac{b_k}{2}p - i, V_{i \bmod p}^k, W_{i \bmod p}^k) & \frac{b_k}{2}(p-1) \leq i \leq \frac{b_k}{2}p - 1 \\ join_k(\lceil \frac{i+1 - \frac{b_k}{2}p}{p-1} \rceil, Z^k, W_{i \bmod p}^k) & \frac{b_k}{2}p \leq i \leq p(b_k-1) - (\frac{b_k}{2} - 1) \end{cases}$$

For example, to create  $X_1^k$  we take the first element of  $V_1^k$  and the rest of elements from  $U_1^k$  obtaining the sequence  $(k-1) \cdot (\pm k)^{p-2} \cdot (-k) \cdot (\pm k)^{p-1} \cdot (-k) \cdot (\pm k)^{p-1} \cdot (-k) \dots$  of length  $b_k/2$ . In the next lemma we claim that  $X_1^k$  really describes the state after the first stage of computation, where input is a balanced 2-flat sequence.

**Lemma 6.** Let  $k \geq p$  and let  $\bar{c} = (c_1, \dots, c_{b_k})$  be a balanced 2-flat sequence of integers from  $[0, 2^{k-1} - 1]$ . Then  $(\hat{Q}_1^k)(reduce(\bar{c})) \in I(X_1^k)$ .

*Proof.* Recall that  $H_i = 2^i - 1$ . Let  $s = height(\bar{c})$  and  $\bar{d} = (d_1, \dots, d_{b_k/2}) = reduce(\bar{c})$ . By Definitions 5 and 6  $s = c_i + c_{b_k-i+1}$  and each  $d_i = c_i - \frac{s}{2} = \frac{c_i - c_{b_k-i+1}}{2}$ . Observe that each  $d_i \in I(\pm k) = [-\frac{H_{k-1}}{2}, \frac{H_{k-1}}{2}]$ . We can get this from the following sequence of inequalities:  $-\frac{H_{k-1}}{2} \leq \frac{-c_{b_k-i+1}}{2} \leq \frac{c_i - c_{b_k-i+1}}{2} \leq \frac{c_i}{2} \leq \frac{H_{k-1}}{2}$ . Moreover, the sequence  $\bar{d}$  is 2-flat, because  $\bar{c}$  is 2-flat. That means that  $d_1 \leq d_3 \leq d_5 \leq \dots \leq d_{k'} \leq d_1 + 1$  and  $d_2 \leq d_4 \leq d_6 \leq \dots \leq d_{k''} \leq d_2 + 1$ , where  $k' = 2\lceil \frac{b_k}{4} \rceil - 1$  and  $k'' = 2\lfloor \frac{b_k}{4} \rfloor$ .

**Fact 12.** Either  $-\frac{1}{2} \leq d_1$  and  $d_{k''} \leq 0$  or  $-\frac{1}{2} \leq d_2$  and  $d_{k'} \leq 0$ .

To prove the fact we consider three cases of the value of  $d_1$ .

**Case  $d_1 \geq 0$ .** In this case we have to prove only that  $d_{k''} \leq 0$ . But it is true since  $d_{k''} = \frac{c_{k''} - c_{b_k-k''+1}}{2} \leq \frac{c_{b_k} - c_1}{2} = -d_1 \leq 0$ . The last inequality holds, because  $\bar{c}$  is 2-flat and both  $k''$  and  $b_k$  are even.

**Case  $d_1 \leq -1$ .** Then  $d_{k'} \leq d_1 + 1 \leq 0$ . Thus it remains to prove that  $d_2 \geq -\frac{1}{2}$ . Similar to the previous case, we observe that  $d_2 = \frac{c_2 - c_{b_k-1}}{2} \geq \frac{c_{b_k-1} - (c_1 + 1)}{2} = -d_1 - 1 \geq 0$ .

**Case  $d_1 = -\frac{1}{2}$ .** Then  $d_{k'} \leq d_1 + 1 = \frac{1}{2}$  and from  $-\frac{1}{2} = \frac{c_1 - c_{b_k}}{2}$  we get  $c_1 + 1 = c_{b_k} \leq c_2 + 1$ . Since  $c_2 \geq c_1$ , we have  $d_2 \geq d_1 = -\frac{1}{2}$ . If  $d_{k'} \leq 0$ , we are done. Otherwise  $d_{k'} = \frac{1}{2}$  and we have to show that  $d_{k''} \leq 0$ . To this end let us notice that  $\frac{s}{2} = c_1 - d_1 = c_1 + \frac{1}{2}$  and  $c_{b_k-k'+1} = s - c_{k'} = s - (d_{k'} + \frac{s}{2}) = \frac{s}{2} - \frac{1}{2} = c_1$ . It follows that  $c_{k''} = c_1$  since  $c_1 \leq c_2 \leq c_{k''} \leq c_{b_k-k'+1} = c_1$ . Thus  $d_{k''} = d_1 = -\frac{1}{2}$  and this concludes the proof of Fact 12.

From Fact 12 and since  $\bar{d}$  is 2-flat we can immediately get the following corollary.

**Corollary 3.**  $\bar{d} \in I((k-1, -k, k-1, -k, \dots) \cup I(-k, k-1, -k, k-1, \dots))$ .

To finish the proof of the lemma we need one more fact:

**Fact 13.**  $(\hat{Q}_1^k)(I((k-1, -k, k-1, -k, \dots) \cup I(-k, k-1, -k, k-1, \dots))) \subseteq I(X_1^k)$ .

Observe, firstly, that the *Cyc* function is applied to the first position in an input sequence, thus the input to *Cyc* is either from  $I(k-1)$  or from  $I(-k)$ . By Fact 11.2,  $Cyc(I(k-1)) \subseteq I(k-1)$  and  $Cyc(-k) \subseteq I(k-1)$ , thus each corresponding output on the first position is correct. On the other positions in the output sequence we have either  $I(\pm k)$  or  $I(-k)$  and  $I(-k)$  appears only on positions, which indices are multiples of  $p$ . If  $j$ ,  $1 \leq j \leq \frac{b_k}{2}$ , is a multiple of  $p$ , then  $j \in args(mov_j^k) \in Q_1^k$  and that means that in  $\hat{Q}_1^k$  the *MinMax* function is applied to positions  $j$  and  $j+1$  or the *Min* function if  $j = \frac{b_k}{2}$ . In the former case, on the positions  $j$  and  $j+1$  in an input sequence, we have a pair from either  $I(-k, k-1)$  or  $I(k-1, -k)$ . By Fact 11.4 the output

	1	2	3	4	...	p-1	p	p+1	p+2	...
Q 1	Cyc	Dec k-1-(p-1)	Dec k-1-2(p-1)	Dec k-1-3(p-1)	...	Dec k-1-(p-2)(p-1)	MinMax	Dec k-0-p(p-1)	...	...
X 1	k-1	±k	±k	±k	...	±k	-k	±k	±k	...
Q 2	Dec k-2	Dec k-2-(p-1)	Dec k-2-2(p-1)	Dec k-2-3(p-1)	...	MinMax	Dec k-1-(p-1)(p-1)	Dec k-1-p(p-1)	...	...
X 2	k-2	±k	±k	±k	...	-k	±k	±k	±k	...
Q 3	Dec k-3	Dec k-3-(p-1)	Dec k-3-2(p-1)	Dec k-3-3(p-1)	...	(←-Min)Max	Dec k-2-(p-2)(p-1)	Dec k-2-(p-1)(p-1)	Dec k-2-p(p-1)	...
X 3	k-3	±k	±k	±k	...	±k	±k	±k	±k	...
Q 4	Dec k-4	Dec k-4-(p-1)	Dec k-4-2(p-1)	Dec k-4-3(p-1)	...	Dec k-3-(p-2)(p-1)	Dec k-3-(p-1)(p-1)	Dec k-3-p(p-1)	...	...
X 4	k-4	±k	±k	±k	...	±k	±k	±k	±k	...
...	...	...	...	...	...	...	...	...	...	...
Q p-1	Dec k-p+1	MinMax	Dec k+1-3(p-1)	...	Dec k+1-(p-2)(p-1)	Dec k+1-(p-1)(p-1)	Dec k+1-p(p-1)	Min(Max→)	...	...
X p-1	k-p+1	-k	±k	±k	...	±k	±k	±k	±k	...
Q p	MinMax	Dec k-0-2(p-1)	Dec k-0-3(p-1)	...	Dec k-0-(p-2)(p-1)	Dec k-0-(p-1)(p-1)	MinMax	...	...	...
X p	-k	k-p+1	±k	±k	...	±k	±k	±k	±k	...
Q 1	Cyc	Dec k-1-(p-1)	Dec k-1-2(p-1)	Dec k-1-3(p-1)	...	Dec k-1-(p-2)(p-1)	MinMax	Dec k-0-p(p-1)	...	...
X p+1	k-1	k-p	±k	±k	...	±k	-k	±k	±k	...
Q 2	Dec k-2	Dec k-2-(p-1)	Dec k-2-2(p-1)	Dec k-2-3(p-1)	...	MinMax	Dec k-1-(p-1)(p-1)	Dec k-1-p(p-1)	...	...
X p+2	k-2	k-p-1	±k	±k	...	-k	±k	±k	±k	...
Q 3	Dec k-3	Dec k-3-(p-1)	Dec k-3-2(p-1)	Dec k-3-3(p-1)	...	(←-Min)Max	Dec k-2-(p-2)(p-1)	Dec k-2-(p-1)(p-1)	Dec k-2-p(p-1)	...
X p+3	k-3	k-p-2	±k	±k	...	±k	±k	±k	±k	...
...	...	...	...	...	...	...	...	...	...	...

Figure 4: The initial steps of a computation on sequences of interval descriptors.

on the  $j$ -th position must be from  $I(-k)$ . In the later case, the *Min* function is applied to an element of  $I(-k) \cup I(k-1) \subseteq I(\pm k)$ . By Fact 11.3 we have  $\text{Min}(I(\pm k)) \subseteq I(-k)$ . Finally, on a position  $j > 1$  such that  $j \bmod p \notin \{0, 1\}$ , the input value is from  $I(\pm k)$  and only a  $\text{Dec}_*$  function can be applied to that value. But  $\text{Dec}_i(I(\pm k)) \subseteq I(\pm k)$  by Fact 11.1 and that finishes the proof of the fact.  $\square$   $\square$

Informally speaking, the next steps of a computation go as follows: each value  $\pm k$  is moving to its neighbour right position every  $p-1$  round with the help of *MinMax* function; at the last position the value is changed to  $-k$  by *Min* function; each value  $-k$  is moving to the left every round and at the first position it is changed to  $k-1$  by *Cyc*; each value  $k-1$  is decreased by one  $p-2$  times at the first position with the help of  $\text{Dec}_*$  functions, then it is moved to the second position, decreased  $p-2$  times again and so on; at the last position the value is finally decreased from one to zero by  $\text{Dec}_0$  or *Min* and starts moving to the left, one position a round, stopping at the first position or next to the previous zero. After  $p(b_k-1) - (\frac{b_k}{2}-1)$  rounds the sequence contains only zeroes. See Figure 4 to observe the initial steps of the process. In the next lemma we formally describe such computations. To prove it we need one more technical fact.

**Fact 14.** For all  $i$  and  $l$  such that  $1 < i \leq p(b_k-1) - (\frac{b_k}{2}-1)$  and  $1 \leq l \leq b_k/2$  the pair  $(X_{i-1,l}^k, X_{i,l}^k)$  is equal to either (a)  $(U_{x',l}^k, U_{x,l}^k)$  or (b)  $(V_{x',l}^k, V_{x,l}^k)$  or (c)  $(W_{x',l}^k, W_{x,l}^k)$  or (d)  $(Z_l^k, Z_l^k)$ , where  $x' = (i-1) \bmod p$  and  $x = i \bmod p$ .

*Proof.* The fact is obviously true for such pair of  $i$  and  $l$  that both  $X_{i-1}^k$  and  $X_i^k$  are defined in the same case of Definition 12 and the first argument of  $\text{join}_k$  does not change its value between  $X_{i-1}^k$  and  $X_i^k$ . Thus we have to prove the fact for the following other cases.

**Case**  $1 < i < \frac{b_k}{2}(p-1)$ . We have to consider only  $i = a(p-1)$  and  $l = a+1$  for an integer  $a$ . Then  $x = i \bmod p \equiv -a \pmod{p}$  and  $x+l \equiv 1 \pmod{p}$ . By Definition 12, we have  $X_{i-1,l}^k = U_{x',l}^k$  and  $X_{i,l}^k = V_{x,l}^k$ . By Definition 10,  $V_{x,l}^k = U_{x,l}^k$ , thus we are in case (a) of the fact.

**Case**  $i = \frac{b_k}{2}(p-1)$ . Then  $X_{i-1}^k = \text{join}_k(b_k/2, V_{x'}^k, U_{x'}^k) = V_{x'}^k$  and  $X_i^k = \text{join}_k(b_k/2, V_x^k, W_x^k) = V_x^k$  by the definition. It follows that for all values of  $l$  we get case (b) of the fact.

**Case**  $\frac{b_k}{2}(p-1) < i < \frac{b_k}{2}p$ . We have to consider only  $l = \frac{b_k}{2}p - i$ . If  $x+l \not\equiv 1 \pmod{p}$  then  $W_{x,l}^k = V_{x,l}^k$ , by Definition 10, and we are again in case (b) of the fact. Otherwise  $x+l \equiv 1 \pmod{p}$ , but  $x'+l \not\equiv 1 \pmod{p}$ . In this case  $X_{i-1}^k = V_{x',l}^k = W_{x',l}^k$  and  $X_i^k = W_{x,l}^k$ . Thus we get case (c) of the fact.

**Case**  $i = \frac{b_k}{2}p$ . Then  $X_{i-1}^k = \text{join}_k(1, V_{x'}^k, W_{x'}^k)$  and  $X_i^k = \text{join}_k(1, Z^k, W_x^k)$  by the definition. The only case we have to check is  $l = 1$ . In this case  $x+l \equiv 1 \pmod{p}$  and, by Definition 10,  $W_{x,1}^k = 0 = Z_1^k = X_{i,1}^k$ . In addition, we have  $x'+l \not\equiv 1 \pmod{p}$ , therefore  $X_{i-1,1}^k = V_{x',1}^k = W_{x',1}^k$  and we have case (c) of the fact.

**Case**  $\frac{b_k}{2}p < i \leq p(b_k-1) - (\frac{b_k}{2}-1)$ . In this last case we have to consider only  $i = \frac{b_k}{2}p + b(p-1)$  and  $l = b+1$  for an integer  $b$ . Then  $x \equiv -b \pmod{p}$  and, consequently,  $x+l \equiv 1 \pmod{p}$ . It follows that  $X_{i,l}^k = Z_l^k = 0 = W_{x,l}^k$  and, by Definition 12,  $X_{i-1,l}^k = W_{x',l}^k$ , so we are again in case (c) of the fact.  $\square$   $\square$

In the next key lemma of this subsections we claim that state sequences defined in Definition 12 really describe any computation on intervals assuming that we start with a balanced 2-flat sequence.

**Lemma 7.** For  $k \geq p$  and each  $i = 1, 2, \dots, p(b_k - 1) - \frac{b_k}{2}$  the following inclusion holds:

$$(\hat{Q}_{i \bmod p+1}^k)(I(X_i^k)) \subseteq I(X_{i+1}^k).$$

*Proof.* We have to prove, equivalently, that for  $k \geq p$  and  $x = 1, \dots, p$  the following inclusions are true:  $(\hat{Q}_x^k)(I(X_{pj+x-1}^k)) \subseteq I(X_{pj+x}^k)$ , where  $j = 1, \dots, \lfloor \frac{p(b_k-1)-b_k/2}{p} \rfloor$  for  $x = 1$  and  $j = 0, 1, \dots, \lfloor \frac{p(b_k-1)-b_k/2-x+1}{p} \rfloor$  for  $x > 1$ . The value of the function  $\hat{Q}_x^k$ ,  $x = 1, \dots, p$ , on a fixed position can be computed with the help of one of the functions  $Cyc$ ,  $Dec_*$ ,  $MinMax$  and  $Min$  introduced in Definition 8 (see also Fact 10). We consider these functions one after another analysing which positions in state sequences are modified by them and what values are in that positions before and after applying a function. In the following, we denote by  $A_{i,j}$  the  $j$ -th element of a sequence  $A_i$ .

The function  $Cyc$  corresponds to  $c\hat{y}c^k$ , which is used only in the definition of  $\hat{Q}_1^k$  and modifies just the position 1 of the sequences  $I(X_{pj}^k)$ , where  $j = 1, \dots, \lfloor \frac{p(b_k-1)-b_k/2}{p} \rfloor$ . Thus it is enough to show the inclusion  $Cyc(I(X_{pj,1}^k)) \subseteq I(X_{pj+1,1}^k)$ . By Definition 12 the argument of  $Cyc \cdot I$  can be either  $X_{pj,1}^k = V_{0,1}^k = -k$  for  $pj < \frac{b_k}{2}p$  or  $X_{pj,1}^k = Z_1 = 0$  for  $pj \geq \frac{b_k}{2}p$ . The corresponding value of the next state sequence is  $X_{pj+1,1}^k = V_{1,1}^k = k-1$  for  $pj+1 < \frac{b_k}{2}p$  or  $X_{pj+1,1}^k = Z_1 = 0$  for  $pj+1 \geq \frac{b_k}{2}p$ . Using Fact 11, both inclusions  $Cyc(I(-k)) \subseteq I(k-1)$  and  $Cyc(I(0)) \subseteq I(0)$  are true and we are done.

In the set  $\hat{Q}_x^k$  there are several  $\hat{d}ec_{l,s}^k$  functions, each of which satisfies the conditions  $x+l \not\equiv 1, 2 \pmod{p}$ ,  $1 \leq l \leq b_k/2$  and  $s = (p-2)(l-1) - 1 - (x+l-1) \bmod p \leq k-1$ . We know also that  $args(\hat{d}ec_{l,s}^k) = \{l\}$  and  $(\hat{d}ec_{l,s}^k(\bar{d}))_l = Dec_{k-s-1}(d_l)$  for a sequence  $\bar{d} = (d_1, \dots, d_{b_k/2})$ , thus we can rewrite our proof goal for that functions as the following fact.

**Fact 15.** Let  $k \geq p$ . For any  $x, l$  and  $s$  such that  $1 \leq x \leq p$ ,  $1 \leq l \leq b_k/2$ ,  $x+l \not\equiv 1, 2 \pmod{p}$  and  $1 \leq s = (p-2)(l-1) - 1 - (x+l-1) \bmod p \leq k-1$  we have

$$Dec_{k-s-1}(I(X_{pj+x-1,l}^k)) \subseteq I(X_{pj+x,l}^k),$$

for any  $j \geq 0$  such that state sequences  $X_{pj+x-1}^k$  and  $X_{pj+x}^k$  are defined.

The sequences  $X_*^k$  are defined with the help of sequences  $U_*^k$ ,  $V_*^k$ ,  $W_*^k$  and  $Z_*^k$ . In  $U_x^k$ ,  $V_x^k$  and  $W_x^k$  there are *strange* “moving-left” elements  $-k$  or  $0$  that appears on positions whose indices  $\equiv -x+1 \pmod{p}$ . Thus those strange elements cannot appear on position  $l$  in  $X_{pj+x-1}^k$  and  $X_{pj+x}^k$ , since, otherwise,  $l \equiv -x+1 \pmod{p}$  or  $l \equiv -(x-1)+1 \pmod{p}$ , but we know that  $l+x \not\equiv 1, 2 \pmod{p}$ . By Fact 14, we have to consider just the following three cases of values  $X_{pj+x-1,l}^k$  and  $X_{pj+x,l}^k$ .

Cases of $y = X_{pj+x-1,l}^k$	Cases of $y' = X_{pj+x,l}^k$	Value of $y$	Value of $y'$
$y = U_{x-1,l}^k$	$y' = U_{x,l}^k$	$\pm k$	$\pm k$
$y = V_{x-1,l}^k = W_{x-1,l}^k$	$y' = V_{x,l}^k = W_{x,l}^k$	$k-s$	$k-s-1$
$y = Z_l$	$y' = Z_l$	$0$	$0$

In all cases above we have  $Dec_{k-s-1}(I(y)) \subseteq I(y')$  by Fact 11.1. Since it is not obvious that  $V_{x-1,l}^k = k-s$  and  $V_{x,l}^k = k-s-1$  (the second case in the table), we prove these equations now. Since  $x+l \not\equiv 1, 2 \pmod{p}$  and  $s \leq k-1$ , it follows that  $V_{x-1,l}^k = e_p^k(x-1, l) = \max(0, k - (p-2)(l-1) - (x-1+l-1) \bmod p) = k - \min(k, (p-2)(l-1) - 1 + (x+l-1) \bmod p) = k-s$ . In a similar way,  $V_{x,l}^k = e_p^k(x, l) = \max(0, k - (p-2)(l-1) - (x+l-1) \bmod p) = k-1 - \min(k-1, -1 + (p-2)(l-1) + (x+l-1) \bmod p) = k-s-1$ .

The next function to be analysed is  $MinMax$ . It corresponds to all  $\hat{m}ov_l^k$  functions in a set  $\hat{Q}_x^k$ , where  $1 \leq l < b_k/2$  and  $1 \leq x \leq p$ . By Definitions 4 and 8, each such function satisfies the condition  $l+x \equiv 1 \pmod{p}$ . We know also that  $args(\hat{m}ov_l^k) = \{l, l+1\}$  and, by Fact 10  $(\hat{m}ov_l^k(\bar{d}))_{l+j} = (MinMax(d_l, d_{l+1}))_{1+j}$  for  $j \in \{0, 1\}$  and any sequence  $\bar{d} = (d_1, \dots, d_{b_k/2})$ . Thus, to prove the lemma, it suffices to show the following fact.

**Fact 16.** Let  $k \geq p$ . For any  $x$  and  $l$  such that  $1 \leq x \leq p$ ,  $1 \leq l < b_k/2$  and  $l + x \equiv 1 \pmod{p}$  we have

$$\text{MinMax}(I(X_{pj+x-1,l}^k, X_{pj+x-1,l+1}^k)) \subseteq I(X_{pj+x,l}^k, X_{pj+x,l+1}^k),$$

where  $j \geq 0$  is an integer such that state sequences  $X_{pj+x-1}^k$  and  $X_{pj+x}^k$  are defined.

As we do with the previous functions, we prove the fact by considering all possible cases in the following table. All of its values are set according to Definition 10, since  $x + l = x - 1 + l + 1 \equiv 1 \pmod{p}$ . To reduce the size of the table we also use the following shortcuts:  $a = pj + x$  and  $y = k - (p - 2)l - 1$ .

Cases of $(s_1, s_2)$		Cases of $(t_1, t_2)$		Value of		Value of	
$s_1 = X_{a-1,l}^k$	$s_2 = X_{a-1,l+1}^k$	$t_1 = X_{a,l}^k$	$t_2 = X_{a,l+1}^k$	$s_1$	$s_2$	$t_1$	$t_2$
$U_{x-1,l}^k$	$U_{x-1,l+1}^k$	$U_{x,l}^k = V_{x,l}^k$	$U_{x,l+1}^k$	$\pm k$	$-k$	$-k$	$\pm k$
$V_{x-1,l}^k$	$V_{x-1,l+1}^k = U_{x-1,l+1}^k$	$V_{x,l}^k$	$V_{x,l+1}^k = W_{x,l+1}^k$	$y$	$-k$	$-k$	$y$
$W_{x-1,l}^k = V_{x-1,l}^k$	$W_{x-1,l+1}^k = Z_{l+1}$	$W_{x,l}^k$	$W_{x,l+1}^k$	$y$	$0$	$0$	$y$
$Z_l$	$Z_{l+1} = W_{x-1,l+1}^k$	$Z_l$	$Z_{l+1}$	$0$	$0$	$0$	$0$

In all cases above we have  $\text{MinMax}(I(s_1, s_2)) \subseteq I(t_1, t_2)$  by Facts 11.4 and 11.5. Thus, to end the proof of the fact we have to check whether  $V_{x-1,l}^k = V_{x,l+1}^k = y$ . From the definition  $V_{x-1,l}^k = e_p^k(x-1, l) = \max(0, k - (p-2)(l-1) - (x-1+l-1) \bmod p) = \max(0, k - (p-2)l + (p-2) - (p-1)) = \max(0, k - (p-2)l - 1) = y$ . The equality  $(x-1+l-1) \bmod p = p-1$  follows from  $x+l \equiv 1 \pmod{p}$ . We use also the fact that from  $l < b_k/2$  we can get  $(p-2)l + 1 \leq k$ . In the same way,  $V_{x,l+1}^k = e_p^k(x, l+1) = \max(0, k - (p-2)l - (x+l+1-1) \bmod p) = y$ .

The last function to be considered is  $\text{Min}$ . It corresponds to all  $\hat{\text{mov}}_l^k$  functions in  $\hat{Q}_x^k$ ,  $1 \leq x \leq p$ , such that  $x + l \equiv 1 \pmod{p}$  and  $l = (k-2)/(p-2)$ . Thus, to finish the proof of the lemma, it suffices to show the following fact.

**Fact 17.** Let  $k \geq p$ . For all  $1 \leq x \leq p$  and  $l = (k-2)/(p-2)$  such that  $x + l \equiv 1 \pmod{p}$  we have

$$\text{Min}(I(X_{pj+x-1,l}^k)) \subseteq I(X_{pj+x,l}^k),$$

where  $j \geq 0$  is an integer such that state sequences  $X_{pj+x-1}^k$  and  $X_{pj+x}^k$  are defined.

As in the case of previous functions we prove the fact by considering all possible cases in the following table.

Cases of	Cases of	Value of	Value of
$s = X_{pj+x-1,l}^k$	$t = X_{pj+x,l}^k$	$s$	$t$
$s = U_{x-1,l}^k$	$t = U_{x,l}^k = V_{x,l}^k$	$\pm k$	$-k$
$s = V_{x-1,l}^k = W_{x-1,l}^k$	$t = W_{x,l}^k = Z_l$	$1$	$0$
$s = Z_l$	$t = Z_l$	$0$	$0$

In all cases above we have  $\text{Min}(I(s)) \subseteq I(t)$  by Fact 11.3. Observe that  $l + x - 1 \equiv 0 \pmod{p}$ , thus we have to check whether  $V_{x-1,l}^k = 1$ . By the definition  $V_{x-1,l}^k = e_p^k(x-1, l) = \max(0, k - (p-2)(l-1) - (x-1+l-1) \bmod p) = \max(0, k - (p-2)l + p - 2 - (p-1)) = \max(0, k - (k-2) - 1) = 1$ .  $\square$   $\square$

**Lemma 8.** Let  $k \geq p$ ,  $D = p(b_k - 1) - (\frac{b_k}{2} - 1)$  and let  $\bar{c} = (c_1, \dots, c_{b_k})$  be a balanced 2-flat sequence of integers from  $[0, 2^{k-1} - 1]$  and let  $s = \text{height}(\bar{c})$ . Let  $f = f_D \circ f_{D-1} \circ \dots \circ f_1$ , where  $f_i = \mathcal{Q}_{((i-1) \bmod 3) + 1}^k$ ,  $i = 1, \dots, D$ . Then  $f(\bar{c}) = (\frac{s}{2})^{b_k}$  if  $s$  is even or  $f(\bar{c}) = (\frac{s-1}{2})^{b_k/2} \cdot (\frac{s+1}{2})^{b_k/2}$  otherwise.

*Proof.* Since each  $f_i$  maps a balanced sequence to a balanced one, let  $\hat{f}_i = \text{reduce}(f_i, s) = \hat{Q}_{((i-1) \bmod p) + 1}^k$ , where the later equality follows from Lemma 5. Let also  $\bar{d}_0 = \text{reduce}(\bar{c})$  and let  $\bar{d}_i = \hat{f}_i(\bar{d}_{i-1})$  for  $i = 1, \dots, D$ . Then  $\bar{d}_1 \in I(X_1^k)$  by Lemma 6 and for  $i = 2, \dots, D$  we get  $\bar{d}_i \in I(X_i^k)$  by an easy induction and Lemma 7. Let  $\mathbb{Z}$  denote, as usual, the set of integers. By  $\mathbb{Z}_{\frac{1}{2}}$  we will denote the set  $\{z + \frac{1}{2} | z \in \mathbb{Z}\}$ . Looking at Definitions 6 and 8 observe the following fact:

**Fact 18.** If  $s$  is even then all elements of sequences  $\bar{d}_i$ ,  $i = 0, \dots, D$ , are integers. If  $s$  is odd then all elements of sequences  $\bar{d}_i$ ,  $i = 0, \dots, D$ , are in  $\mathbb{Z}_{\frac{1}{2}}$ .

Since  $\bar{d}_D \in I(X_D^k) = I(0^{b_k/2})$  and  $I(0) \cap \mathbb{Z} = \{0\}$  and  $I(0) \cap \mathbb{Z}_{\frac{1}{2}} = \{\frac{1}{2}\}$ , it follows that  $\bar{d}_D = 0^{b_k/2}$  if  $s$  is even and  $\bar{d}_D = \frac{1}{2}^{b_k/2}$ , otherwise. Using now the definition of  $s$ -extended sequence to  $0^{b_k/2}$  and  $\frac{1}{2}^{b_k/2}$  we get the desired conclusion of the lemma.  $\square$   $\square$

In this way, with respect to Lemma 3, we have proved that the network  $M_k$  is able to merge in  $p(b_k - 1) - (\frac{b_k}{2} - 1)$  stages two sorted sequences given in odd and even registers, provided that the numbers of ones in our matrix columns form a balanced sequence. If the sequence is not balanced,  $\frac{b_k}{2} - 1$  additional stages are needed to get a sorted output.

### 3.3 Analysis of General Columns

In a general case we will use balanced sequences as lower and upper bounds on the numbers of ones in our matrix columns and observe that  $Q_x^k$ ,  $1 \leq x \leq p$ , are monotone functions (see Fact 8).

**Definition 13.** Let  $k \geq p$  and let  $\bar{c} = (c_1, \dots, c_{b_k})$  be a 2-flat sequence of integers from  $[0, 2^{k-1} - 1]$  that is not balanced. Since both  $\bar{c}_{odd} = (c_1, \dots, c_{b_k-1})$  and  $\bar{c}_{evn} = (c_2, \dots, c_{b_k})$  are flat sequences, let  $i$  ( $j$ , respectively) be such that  $c_{2i-1} < c_{2i+1}$  ( $c_{b_k-2j} < c_{b_k-2j+2}$ , respectively) or let  $i = k-2$  ( $j = k-2$ ) if  $\bar{c}_{odd}$  ( $\bar{c}_{evn}$ , respectively) is a constant sequence. The defined below sequences  $\check{c}$  and  $\hat{c}$  we will call lower and upper bounds of  $\bar{c}$ . If  $i < j$  then for  $l = 1, \dots, b_k$

$$\check{c}_l = \begin{cases} c_1 & \text{if } l \text{ is odd and } l \leq 2j-1 \\ c_{b_k-1} & \text{if } l \text{ is odd and } l \geq 2j+1 \\ c_l & \text{if } l \text{ is even} \end{cases} \quad \hat{c}_l = \begin{cases} c_{b_k-1} & \text{if } l \text{ is odd} \\ c_{b_k} & \text{if } l \text{ is even} \end{cases}$$

If  $i > j$  then for  $l = 1, \dots, b_k$

$$\check{c}_l = \begin{cases} c_1 & \text{if } l \text{ is odd} \\ c_2 & \text{if } l \text{ is even} \end{cases} \quad \hat{c}_l = \begin{cases} c_l & \text{if } l \text{ is odd} \\ c_2 & \text{if } l \text{ is even and } l \leq b_k - 2i \\ c_{b_k} & \text{if } l \text{ is even and } l > b_k - 2i \end{cases}$$

**Fact 19.** For  $k \geq p$  and any not balanced 2-flat sequence  $\bar{c} = (c_1, \dots, c_{b_k})$  of integers from  $[0, 2^{k-1} - 1]$  the sequences  $\check{c}$  and  $\hat{c}$  are balanced,  $\text{height}(\check{c}) + 1 = \text{height}(\hat{c})$  and  $\check{c} \leq \bar{c} \leq \hat{c}$ .

*Proof.* Let  $i$  and  $j$  be defined as in Definition 13. We will consider only the case  $i < j$ . The proof of the other case is similar. Directly from the definition we get that  $\hat{c}$  is balanced. To see that  $\check{c}$  is also balanced let us check for  $l = 1, \dots, b_k/2$  whether the sum  $\check{c}_{2l-1} + \check{c}_{b_k-2l+2}$  is constant.

$$\check{c}_{2l-1} + \check{c}_{b_k-2l+2} = \check{c}_{2l-1} + c_{b_k-2l+2} = \begin{cases} c_1 + c_{b_k-2l+2} = c_1 + c_{b_k} & \text{if } l \leq j \\ c_{b_k-1} + c_{b_k-2l+2} = c_{b_k-1} + c_2 & \text{otherwise} \end{cases}$$

If  $j = k-2$  there is no otherwise case and we are done. If  $j < k-2$  then  $c_{b_k} - c_2 = c_{b_k-1} - c_1 = 1$ , because of the definition of  $i$  and  $j$  and we are also done. Moreover  $\text{height}(\check{c}) + 1 = c_1 + c_{b_k} + 1 = c_{b_k-1} + c_{b_k} = \text{height}(\hat{c})$ . To prove that  $\check{c} \leq \bar{c} \leq \hat{c}$  we consider even and odd indices. For even indices from the definition we have:  $\check{c}_{2l} = c_{2l} \leq c_{b_k} = \hat{c}_{2l}$ . For odd indices  $\hat{c}_{2l-1} = c_{b_k-1} \geq c_{2l-1} \geq c_1$ . If  $l \leq j$  we are done, otherwise,  $c_{2l-1} = c_{b_k-1} = \check{c}_{2l-1}$ , because  $\bar{c}_{odd}$  is flat.  $\square$   $\square$

**Theorem 2.** Let  $k \geq p$ ,  $D = p(b_k - 1)$  and let  $\bar{c} = (c_1, \dots, c_{b_k})$  be a 2-flat sequence of integers from  $[0, 2^{k-1} - 1]$ . Let  $f = f_D \circ f_{D-1} \circ \dots \circ f_1$ , where  $f_i = Q_{((i-1) \bmod 3)+1}^k$ ,  $i = 1, \dots, D$ . Then  $f(\bar{c})$  is a flat sequence.



*Proof.* Let  $\bar{c}$  be a 2-flat sequence of integers from  $[0, 2^{k-1} - 1]$ . If  $\bar{c}$  is balanced then  $f(\bar{c})$  is a flat sequence due to Lemma 8 and an observation that flat sequences are not modified by  $Q_x^k$  functions. Otherwise, let  $\check{c}$  and  $\hat{c}$  be its balanced lower and upper bounds, as defined in Definition 13. Let  $\bar{c}_0 = \bar{c}$ ,  $\check{c}_0 = \check{c}$ ,  $\hat{c}_0 = \hat{c}$  and for  $i = 1, \dots, D$  let us define  $\bar{c}_i = f_i(\bar{c}_{i-1})$ ,  $\check{c}_i = f_i(\check{c}_{i-1})$  and  $\hat{c}_i = f_i(\hat{c}_{i-1})$ . Observe that  $\check{c}_i \leq \bar{c}_i \leq \hat{c}_i$ , because of monotonicity of functions  $Q_x^k$ ,  $1 \leq x \leq p$ , and Fact 19. To prove that  $\bar{c}_D$  is a flat sequence we need the following three technical facts.

**Fact 20.** *Let  $s = \text{height}(\check{c})$ . If  $s$  is even then  $\bar{c}_{i,j} = \frac{s}{2}$  and  $\bar{c}_{i,b_k-j+1} \in \{\frac{s}{2}, \frac{s}{2} + 1\}$  for each  $i$  and  $j$  such that  $pb_k/2 \leq i \leq D - (b_k/2 - 1)$  and  $1 \leq j \leq \lceil \frac{i+1-(pb_k/2)}{2} \rceil$ . If  $s$  is odd then  $\bar{c}_{i,j} \in \{\frac{s-1}{2}, \frac{s+1}{2}\}$  and  $\bar{c}_{i,b_k-j+1} = \frac{s+1}{2}$  for each  $i$  and  $j$  such that  $pb_k/2 \leq i \leq D - (b_k/2 - 1)$  and  $1 \leq j \leq \lceil \frac{i+1-(pb_k/2)}{2} \rceil$ .*

*Proof.* Since both  $\check{c}$  and  $\hat{c}$  are balanced, we can consider reduced forms of them and use Lemmas 6 and 7. For the given range of  $i$ 's values that means that both  $\text{reduce}(\check{c}_i)$  and  $\text{reduce}(\hat{c}_i)$  are in  $I(X_i^k)$ . Recall that  $I(X_i^k) = I(\text{join}_k(\lceil \frac{i+1-(pb_k/2)}{2} \rceil, Z^k, W_i^k))$  for  $i = pb_k/2, \dots, D - (b_k/2 - 1)$ . Hence for a given range of  $j$ 's values both  $\text{reduce}(\check{c}_i)_j$  and  $\text{reduce}(\hat{c}_i)_j$  are in  $I(0) = [-\frac{1}{2}, 0]$ . From Fact 19 we know that  $\text{height}(\hat{c}) = s + 1$  and from Lemma 4 that heights are preserved in sequences  $\check{c}_i$  and  $\hat{c}_i$ . Thus, from the definition of a reduced sequence,  $\check{c}_{i,j} \in [\frac{s-1}{2}, \frac{s}{2}]$ ,  $\check{c}_{i,b_k-j+1} \in [\frac{s}{2}, \frac{s+1}{2}]$ ,  $\hat{c}_{i,j} \in [\frac{s}{2}, \frac{s+1}{2}]$  and  $\hat{c}_{i,b_k-j+1} \in [\frac{s+1}{2}, \frac{s+2}{2}]$ . Since all  $\check{c}_i$  and  $\hat{c}_i$  are sequences of integers, for even  $s$  we get  $\check{c}_{i,j} = \check{c}_{i,b_k-j+1} = \hat{c}_{i,j} = \frac{s}{2}$  and  $\hat{c}_{i,b_k-j+1} = \frac{s+2}{2}$ ; for odd  $s$  we conclude that  $\check{c}_{i,j} = \frac{s-1}{2}$  and  $\check{c}_{i,b_k-j+1} = \hat{c}_{i,j} = \hat{c}_{i,b_k-j+1} = \frac{s+1}{2}$ . Since  $\check{c}_{i,j} \leq \bar{c}_{i,j} \leq \hat{c}_{i,j}$ , the fact follows.  $\square$   $\square$

The second fact extends the first fact up to the last stage of our computation.

**Fact 21.** *Let  $s = \text{height}(\check{c})$ . If  $s$  is even then  $\bar{c}_{i,j} = \frac{s}{2}$  and  $\bar{c}_{i,b_k-j+1} \in \{\frac{s}{2}, \frac{s}{2} + 1\}$  for each  $i = D - (b_k/2 - 2), \dots, D$  and  $j = 1, \dots, b_k/2$ . If  $s$  is odd then  $\bar{c}_{i,j} \in \{\frac{s-1}{2}, \frac{s+1}{2}\}$  and  $\bar{c}_{i,b_k-j+1} = \frac{s+1}{2}$  for each  $i = D - (b_k/2 - 2), \dots, D$  and  $j = 1, \dots, b_k/2$ .*

*Proof.* Consider first the sequence  $\bar{c}_{D-(b_k/2-1)}$  and observe that for  $i = D - (b_k/2 - 1)$  the value of  $\lceil \frac{i+1-(pb_k/2)}{p-1} \rceil$  is equal to  $b_k/2$ . It follows from Fact 20 that for even  $s$  all values from the left half of  $\bar{c}_{D-(b_k/2-1)}$  are equal to  $\frac{s}{2}$  and all values from the right half of  $\bar{c}_{D-(b_k/2-1)}$  are in  $\{\frac{s}{2}, \frac{s}{2} + 1\}$ . For odd  $s$  all values from the left half of  $\bar{c}_{D-(b_k/2-1)}$  are in  $\{\frac{s-1}{2}, \frac{s+1}{2}\}$  and all values from the right half of  $\bar{c}_{D-(b_k/2-1)}$  are equal to  $\frac{s+1}{2}$ . Since  $Q_x^k$ ,  $1 \leq x \leq p$ , are built of functions  $\text{dec}_*^k$ ,  $\text{mov}_*^k$  and  $\text{cyc}^k$  (cf. Definitions 3 and 4) observe that each function  $f_i$ ,  $i = D - (b_k/2 - 2), \dots, D$  can exchange only the values at positions from  $\text{args}(\text{mov}_*^k)$  that are from non-constant half of arguments (in case of  $\text{dec}_*^k$  and  $\text{cyc}^k$  we can observe that for  $a \leq b \leq a + 1$  and any  $h \geq 0$  we have  $\min(a, b + h) = a$ ,  $\max(a - h, b) = b$ ,  $\max(a, b - 1) = a$  and  $\min(a + 1, b) = b$ , that is, the functions are identity mappings in stages  $D - (b_k/2 - 2), \dots, D$ ). The  $\text{mov}_*^k$  functions can exchange only unequal values at neighbor positions moving the smaller value to the left.  $\square$   $\square$

The last fact states that unequal values  $\bar{c}_{i,j}$  described in the previous two facts are getting sorted during the last stages of the computation. Observe that if  $s$  is odd (even, respectively) then we have to trace the sorting process only in a left (right, respectively) region of indices  $[1, \min(b_k/2, \lceil \frac{i+1-(pb_k/2)}{p-1} \rceil)]$  ( $[\max(b_k/2 + 1, b_k - \lceil \frac{i+1-(pb_k/2)}{p-1} \rceil + 1), b_k]$ , respectively), where  $i = pb_k/2, \dots, D$  and the values to be sorted differs at most by one. The other part is already sorted. We trace the positions of the smaller values  $s' = \frac{s-1}{2}$  in the left region and the greater values  $s' = \frac{s}{2} + 1$  in the right region. We will call each such  $s'$  a moving element. For  $t = 1, \dots, b_k/2$  let us define  $i_t = pb_k/2 + (p-1)(t-1)$  to be the stage, after which the length of the region extends from  $t-1$  to  $t$  and a new element appears in it. Let  $t' = t$  for odd  $s$  and  $t' = b_k - t + 1$ , otherwise, be the position of this new element and  $a_t = c_{i_t, t'}$  be its value. Finally, let  $n_t = |\{1 \leq l \leq t \mid a_l = s'\}|$  be the number of moving elements in the region after stage  $i_t$ .

**Fact 22.** *Using the above definitions, for  $t = 1, \dots, b_k/2$ , if  $a_t = s'$  then for  $i = 0, \dots, D - i_t$  we have  $c_{i_t+i, \max(t-i, n_t)} = a_t$  if  $s$  is odd and  $c_{i_t+i, \min(t'+i, b_k-n_k+1)} = a_t$ , otherwise.*

*Proof.* We prove the fact only for odd  $s$ , that is, for the left region. The proof for the right region is symmetric. We would like to show that if  $a_t = s'$  appears at position  $t' = t$  after stage  $i_t$  then it moves in each of the following stages one position to the left up to its final position  $n_t$ . The proof is by induction on  $t$  and  $i$ . If  $t = 1$  and  $a_1 = s'$  appears at position 1 after stage  $i_1 = pb_k/2$  then  $n_1 = 1$  and  $a_1$  is already at its final

position. It never moves, because values at second position are  $\geq s'$ , by Facts 20 and 21. If  $t > 1$  and  $a_t = s'$  then the basis  $i = 0$  is obviously true. In the inductive step  $i > 0$  we assume that  $c_{i+i-1, \max(t'-i+1, n_t)} = a_t$  and that the fact is true for smaller values of  $t$ . If  $\max(t-i+1, n_t) = n_t$  then also  $\max(t-i, n_t) = n_t$  and, by the induction hypothesis, values at positions  $1, \dots, n_t - 1$  are all equal  $s'$ . That means that  $a_t$  is at its final position and we are done. Thus we left with the case:  $n_t < t - i + 1$ , that is, with  $n_t \leq t - i$ .

Consider the sequences  $\bar{c}_{i+i-1}$  and  $\bar{c}_{i+i} = f_{i+i}(\bar{c}_{i+i-1})$ . We know that  $\bar{c}_{i+i-1, t-i+1} = s'$ . To prove that  $\bar{c}_{i+i, t-i} = s'$  we would like to show that  $s'$  is moved one position to the left by  $f_{i+i}$ , i.e. that  $\bar{c}_{i+i-1, t-i} = s' + 1$  and  $\text{mov}_{t-i}^k \in f_{i+i}$ . The later is a direct consequence of an observation that  $\text{mov}_a^k \in f_b$  if and only if  $(a+b) \equiv 1 \pmod{p}$ . In our case  $(t-i) + (i+i) = t+i_t = t + pb_k/2 + (p-1)(t-1) = p(b_k/2 + t-1) + 1 \equiv 1 \pmod{p}$ . To prove the former, let us consider any  $a_u = s'$ ,  $u \leq t-1$ . Then  $i_u \leq i_t - (p-1) \leq i_t - 2$  and  $n_u \leq n_t - 1$ . By the induction hypothesis,  $c_{i_u+j, \max(u-j, n_u)} = s'$ . Setting  $j = i_t - i_u + i - 1$  we get  $j \geq i+1$  and  $\max(u-j, n_u) \leq \max(t-1-(i+1), n_t-1) < \max(t-i, n_t) = t-i$ . Moreover,  $i_u + j = i_t + i - 1$ . That means that in the sequence  $\bar{c}_{i+i-1}$  none of  $n_t$  elements  $s'$  is at position  $t-i$  and, consequently,  $\bar{c}_{i+i-1, t-i} = s' + 1$ . Since  $\text{mov}_{t-i}^k$  switches  $s'$  with  $s' + 1$ , this completes the proof of Fact 22.  $\square$   $\square$

Now we are ready to prove that  $\bar{c}_D$  is a flat sequence. By Fact 21, if  $s$  is odd then  $\bar{c}_D \in \{\frac{s-1}{2}, \frac{s+1}{2}\}^{b_k/2} (\frac{s+1}{2})^{b_k/2}$ , otherwise,  $\bar{c}_D \in (\frac{s}{2})^{b_k/2} \{\frac{s}{2}, \frac{s}{2} + 1\}^{b_k/2}$ . The number of minority (moving) elements in  $\bar{c}_D$  has been denote by  $n_{b_k/2}$ . If  $s$  is odd and  $a_t, t = 1, \dots, b_k/2$ , is a minority element  $\frac{s-1}{2}$ , then, by Fact 22,  $c_{D, n_t} = \frac{s-1}{2}$ . If  $s$  is even and  $a_t, t = 1, \dots, b_k/2$ , is a minority element  $\frac{s}{2} + 1$ , then, by Fact 22,  $c_{D, b_k - n_t + 1} = \frac{s}{2} + 1$ . In both cases this proves that  $\bar{c}_D$  is flat, which completes the proof of Theorem 2.  $\square$   $\square$

### 3.4 Proof of Theorem 1

Theorem 1 follows directly from Theorem 2 and Lemma 3. Let  $k \geq p \geq 4$ ,  $b_k = 2\lceil \frac{k-2}{p-2} \rceil$  and  $\bar{c}$  be any 2-flat sequence of integers from  $[0, 2^{k-1} - 1]$ . By Theorem 2 the result of application  $(Q_p^k \circ Q_{p-1}^k \circ \dots \circ Q_1^k)^{b_k-1}$  to  $(\bar{c})$  is a flat sequence. Then, by Lemma 3, the network  $M_k$  is a  $b_k - 1$ -pass merger of two sorted sequences given in odd and even registers, respectively.

## 4 Average sorting times

It is easy to observe that our  $M_k^p$  networks are also periodic sorters, because they contain all neighbour comparators  $[i : i+1]$ ,  $1 \leq i < N_k^p$ . We were curious how efficient periodic sorters were they, when the early stopping property would be applied, that is, when a periodic application of  $M_k^p$  would be stopped just after none of comparator in  $M_k^p$  exchange values. We measured the average and maximal sorting times (the number of rounds) of  $10^5$  (pseudo)random permutations on selected  $M_k^p$  networks,  $3 \leq p \leq 5$ ,  $9 \leq k \leq 14$  and the results are shown in Fig. 5. Surprisingly, the average and maximal sorting times are quite close to  $\log^2(N_k^p)$ . An open question is what is the worst-case sorting time of  $M_k^p$ .

## 5 Conclusions

For each  $k \geq p \geq 4$  we have shown a construction of a  $p$ -periodic merging comparator network of  $N_k^p = (2^k - 2)\lceil \frac{k-2}{p-2} \rceil$  registers and proved that it merge any two sorted sequences (given in odd and even registers, respectively) in time  $D_k^p = p(b_k - 1) = p(2\lceil \frac{k-2}{p-2} \rceil - 1)$ . The construction is regular and quite simple. It is created based on the duality between constant-periodic and constant-delay comparator networks and can be considered as a natural extension of the previous construction of 3-periodic merging networks. Also the proof is a generalisation of the corresponding proof given for 3-periodic merging networks. An open question remains whether the given merging times are optimal for  $p$ -periodic comparator networks.

Finally, one can observe that for  $k > p \geq 4$  we get  $2^k \leq 2(2^k - 2) \leq N_k^p$ , which implies  $k \leq \log N_k^p$ . Now we can bound merging times  $D_k^p$  for  $p = 4, 5, 6$  as  $D_k^4 \leq 4k - 8 \leq 4\log N_k^4$ ,  $D_k^5 \leq 3.33\log N_k^5$  and  $D_k^6 \leq 3\log N_k^6$ . Because of skipped negative terms, exact ratios to  $\log N_k^p$  are even better for small values of  $k$  (compare Fig. 6).

$p$	$k$	$N_k^p$	comparators	$avg(rounds)$	$max(rounds)$	$\frac{avg(rounds)}{\log^2(N_k^p)}$	$\frac{max(rounds)}{\log^2(N_k^p)}$
5	9	1530	3067	103,93	125	92,85%	111,68%
4	9	2040	3577	107,36	124	88,81%	102,58%
5	10	3066	6651	117,04	135	87,25%	100,64%
3	9	3570	5107	133,82	153	96,08%	109,85%
4	10	4088	7673	122,26	136	84,91%	94,49%
5	11	6138	14331	131,58	150	83,10%	94,73%
3	10	8176	11761	165,41	183	97,92%	108,33%
4	11	10230	18423	157,79	176	88,93%	99,19%
5	12	16376	34809	174,96	199	89,27%	101,54%
3	11	18414	26607	200,62	222	99,94%	110,59%
4	12	20470	38903	176,94	196	86,27%	95,56%
5	13	32760	73721	193,31	215	85,92%	95,56%
3	12	40940	59373	239,81	267	102,16%	113,74%
4	13	49140	90101	216,94	240	89,32%	98,81%
5	14	65528	155641	214,66	230	83,85%	89,85%
3	13	90090	131051	283,03	309	104,48%	114,06%
4	14	98292	188405	243,01	264	88,35%	95,98%
3	14	196584	286697	331,11	363	107,08%	117,39%

Figure 5: The average sorting times of  $10^5$  random permutations with  $M_k^p$  networks for  $3 \leq p \leq 5$  and  $9 \leq k \leq 14$ .

$p$	$k$	$N_k^p$	rounds	$\frac{rounds}{\log(N_k^p)}$
6	10	2044	18	163,68%
4	10	4088	28	233,39%
5	11	6138	25	198,67%
4	12	20470	36	251,38%
6	14	49146	30	192,50%
5	14	65528	35	218,75%
4	14	98292	44	265,30%

Figure 6: The merging times of some  $M_k^p$  periodic networks compared to  $\log N$  non-periodic merging time.

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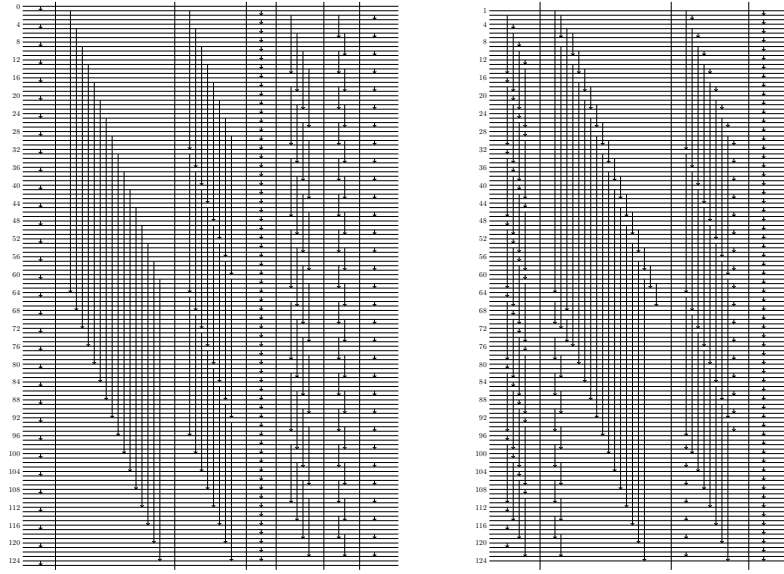


Figure 7: The traditional drawing of  $P_6^4$  (left) and  $M_6^4$  (right) networks. Vertical lines separate stages.

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